5.4 Emittance

The emittance, like the bunch length, is one of the most important parameters of the gun beam, for it helps to determine the brightness of the beam and hence its suitability for applications, such as FELs, requiring high brightness. In this section, I discuss the measurements I did of the emittance of the RF gun beam, and compare these with expectations based on simulations. Ideally, one would like to measure the emittance as a function of beam current and momentum over the entire range of gun operation. Unfortunately, this did not prove possible, and emittance data is only available for a rather limited range of gun operating conditions. The reasons for this will become clear as I proceed. It is necessary first to discuss the principle of the measurements.

A number of methods are used for measuring emittance; a review of some of the commonly-used methods appears in one of the references[11]. The method I employed made use of existing hardware, and involved the variation of a quadrupole upstream of a beam-profile measuring device, in this case a phosphorescent screen. By measuring the beam-size at the screen as a function of the quadrupole strength, one can deduce the RMS geometric emittance, defined by equations (5.42) and (5.43) below. (Note that when I use the word "emittance" in this section, I mean the RMS *geometric* emittance, unless otherwise stated.)

5.4.1 Principle of the Emittance Measurements

The theory of the method as it is usually developed (see, for example, [83]) assumes a mono-energetic beam described by the so-called Σ-matrix[84], which for the x-plane is

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} \mathrm{x}_{\mathrm{rms}}^2 & \langle \mathrm{xx}' \rangle \\ \langle \mathrm{x}'\mathrm{x} \rangle & \mathrm{x}'^2_{\mathrm{rms}} \end{pmatrix}. \tag{5.42}$$

(Often, the beam is assumed to be gaussian, and hence completely characterized by the Σ-matrix, but this assumption is unnecessary, as discussed in subsection 5.4.6.) The RMS geometric emittance in terms of the Σ-matrix is simply

$$\epsilon = \sqrt{\det \mathbf{\Sigma}} = \sqrt{\Sigma_{11}\Sigma_{22} - \Sigma_{12}^2}, \tag{5.43}$$

where I have used the fact that $\Sigma_{12} = \Sigma_{21}$.

An arbitrary Σ-matrix, $\mathbf{\Sigma_0}$, is transformed by a beamline with a transport matrix $\mathbf{r}$ according to:

$$\mathbf{\Sigma_1} = \mathbf{r}\mathbf{\Sigma_0}\mathbf{r}^{\mathrm{T}}, \tag{5.44}$$

where $\mathbf{\Sigma_1}$ is the Σ-matrix at the end of the beamline and where $\mathbf{r}^{\mathrm{T}}$ is the transpose of $\mathbf{r}$. The spatial beam sigma at the end of the beamline—e.g., at a phosphorescent screen—is given by

$$\sigma = \sqrt{(\mathbf{\Sigma_1})_{11}}. \tag{5.45}$$

Clearly, σ is a function of the original Σ-matrix, $\mathbf{\Sigma_0}$, and the matrix $\mathbf{r}$, that is, of the initial beam phase-space and the properties of the beam-transport system between points 0 and 1.

In the experiment, one varies a beamline element—e.g., the strength of a quadrupole—thus obtaining a series of n matrices $\mathbf{r}_{\mathrm{i}}$. Corresponding to each of these matrices is a spatial sigma, σ_{i}, at the end of the beamline. By explicitly writing out equations (5.44) and (5.45), one can express the results of the whole series as

$$\begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_{\mathrm{n}}^2 \end{pmatrix} = \begin{pmatrix} \mathrm{r}_{1,11}^2 & 2\cdot \mathrm{r}_{1,11}\cdot \mathrm{r}_{1,12} & \mathrm{r}_{1,12}^2 \\ \mathrm{r}_{2,11}^2 & 2\cdot \mathrm{r}_{2,11}\cdot \mathrm{r}_{2,12} & \mathrm{r}_{2,12}^2 \\ \vdots & \vdots & \vdots \\ \mathrm{r}_{\mathrm{n},11}^2 & 2\cdot \mathrm{r}_{\mathrm{n},11}\cdot \mathrm{r}_{\mathrm{n},12} & \mathrm{r}_{\mathrm{n},12}^2 \end{pmatrix} \times \begin{pmatrix} \Sigma_{11} \\ \Sigma_{12} \\ \Sigma_{22} \end{pmatrix} \tag{5.46}$$

or

$$\mathbf{M} \;=\; \mathbf{T}\times\mathbf{S}. \tag{5.47}$$

From this, one obtains a solution for the elements of the Σ-matrix:

$$\mathbf{S} = (\mathbf{T}^{\mathrm{T}}\mathbf{T})^{-1}\mathbf{T}^{\mathrm{T}}\mathbf{M} \tag{5.48}$$

from which the emittance is obtained by (see equation (5.43))

$$\epsilon = \sqrt{\mathbf{S}_1\mathbf{S}_3 - \mathbf{S}_2^2} \tag{5.49}$$

Note that this procedure gives not only the emittance, but also the beam-correlations.

5.4.2 Inclusion of Experimental Errors

For experimental data, the σ_i are known within some uncertainty $\Delta\sigma_\mathrm{i}$. In this case, one can use a weighted least-squares fit[85, 61] instead of the equal-weights fit given by equation (5.48), obtaining

$$\mathbf{S} = (\mathbf{T}^\mathrm{T}\mathrm{C}^{-1}\mathbf{T})^{-1}\mathbf{T}^\mathrm{T}\mathrm{C}^{-1}\mathbf{M}, \tag{5.50}$$

where $\mathbf{C}$ is the covariance matrix of the experimentally measured quantities, defined by

$$\mathbf{C}_\mathrm{ij} = \delta_\mathrm{ij}\Delta\sigma_\mathrm{i}\Delta\sigma_\mathrm{j}, \tag{5.51}$$

with δ_ij being the Kronecker delta-function, and where I use Δ to indicate the uncertainties to avoid confusing multiple uses of the letter σ. The uncertainties in the elements of $\mathbf{S}$ (i.e., in the Σ-matrix derived from the fit) are found by computing the covariance matrix of the fit parameters:

$$\mathbf{K} = \mathbf{U}^\mathrm{T}\mathbf{C}\mathbf{U}, \tag{5.52}$$

where $\mathbf{U}$ is defined as

$$\mathbf{U} = (\mathbf{T}^\mathrm{T}\mathrm{C}^{-1}\mathbf{T})^{-1}\mathbf{T}^\mathrm{T}\mathrm{C}^{-1}. \tag{5.53}$$

The uncertainties in the elements of $\mathbf{S}$ are related to the diagonal elements of $\mathbf{K}$, by

$$\Delta\mathbf{S}_\mathrm{i} = \sqrt{\mathbf{K}_\mathrm{ii}} \tag{5.54}$$

The uncertainty in the emittance is then given by propagation of errors through equation (5.49), which, when expressed in terms of the $\Sigma-$matrix yields

$$\begin{aligned}\Delta\epsilon \;=\; & \left(\frac{\Sigma_{11}}{2\epsilon}\right)^2 \mathrm{K}_{33} + \left(\frac{\Sigma_{22}}{2\epsilon}\right)^2 \mathrm{K}_{11} + \left(\frac{\Sigma_{12}}{2\epsilon}\right)^2 \mathrm{K}_{22} + \\ & 2\frac{\Sigma_{11}\Sigma_{22}}{4\epsilon^2}\mathrm{K}_{13} + 2\frac{\Sigma_{11}\Sigma_{12}}{2\epsilon^2}\mathrm{K}_{12} + 2\frac{\Sigma_{12}\Sigma_{22}}{2\epsilon^2}\mathrm{K}_{12}.\end{aligned} \tag{5.55}$$

Note that I have used the off-diagonal terms of the covariance matrix in an attempt to obtain better accuracy. As I will discuss below, this sort of error analysis gives dubious results and I have used a Monte-Carlo method[61] instead for my final results.

5.4.3 Thin Lens Treatment

It is instructive to write equation (5.48) out for the case of a thin lens quadrupole followed by a drift space of length L. In this case, the transport matrix is given by

$$\mathrm{r} = \begin{pmatrix} 1 & \mathrm{L} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\kappa & 1 \end{pmatrix}, \tag{5.56}$$

where $1/\kappa$ is the focal length of the quadrupole. The beam-size at the end of the drift space as a function of the quadrupole focal-length is (from one line of equation (5.46))

$$\sigma^2 = (1-\kappa \mathrm{L})^2 \Sigma_{1,11} + 2\mathrm{L}(1-\kappa\mathrm{L})\Sigma_{1,12} + \mathrm{L}^2\Sigma_{1,22}. \tag{5.57}$$

As pointed out by Ross, *et. al.*,[86], this can be written in a more transparent form as

$$\sigma^2 = \sigma_{\mathrm{o}}^2 \mathrm{L}^2 (\kappa - \kappa_{\mathrm{m}})^2 + \frac{\mathrm{L}^2\epsilon^2}{\sigma_{\mathrm{o}}^2}, \tag{5.58}$$

where σ_{o} is the beam-size at the entrance to the quadrupole and where

$$\kappa_{\mathrm{m}} = \frac{\Sigma_{1,12}\mathrm{L} + \Sigma_{1,11}}{\Sigma_{1,11}\mathrm{L}} \tag{5.59}$$

is the value of κ that gives the minimum spot-size at the screen.

Since σ_{o} is presumably unknown, one must find both the second derivative and the minimum of σ^2 as a function of κ in order to find the emittance. If one rewrites equation (5.58) as

$$\sigma^2 = \mathrm{D}^2(\kappa - \kappa_{\mathrm{m}})^2 + \sigma_{\mathrm{m}}^2, \tag{5.60}$$

where

$$\mathrm{D} = \sqrt{\frac{1}{2}\left(\frac{\partial^2}{\partial \kappa^2}\sigma^2\right)} \tag{5.61}$$