Symplectic Integration in `elegant`

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Michael Borland

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Accelerator Systems Division, Advanced Photon Source

1 Introduction

The program `elegant` [1] performs symplectic integration for hard-edge dipoles and multipoles using the exact Hamiltonian. This was implemented many years ago and never written down, but there are often questions about it. Hence, we've recorded the method in this note. This method is not original but was cobbled together from studying several sources, e.g., [2] and [3].

2 Hamiltonian and Definitions

The exact Hamiltonian for a combined function sector bend is[2]

$$\mathcal{H} = -\frac{eA_s(x,y)}{c} - (1+h_0x)\sqrt{\frac{E^2}{c^2} - m^2c^2 - p_x^2 - p_y^2}, \tag{1}$$

where A_s is the scalar magnetic potential, E is the energy, h_0 is the design curvature of the magnet, p_x is the transverse horizontal momentum, and p_y is the transverse vertical momentum. Defining $(1+\delta) = p/p_0$, $q_x = p_x/p_0$, and $q_y = p_y/p_0$, we have

$$H = H_f + H_d \tag{2}$$

where

$$H_f = -\frac{eA_s(x,y)}{p_0} \tag{3}$$

is the part of the Hamiltonian pertaining to fields and

$$H_d = -(1+h_0x)\sqrt{(1+\delta)^2 - q_x^2 - q_y^2}. \tag{4}$$

is the Hamiltonian for a generalized drift (possibly in curvilinear coordinates if $h_0 \neq 0$).

3 Integration for Dipole

The motion will be integrated using a drift-kick-drift technique (perhaps in higher order). For a drift, Hamilton's equations for the momenta are

$$\frac{dq_x}{ds} = -\frac{\partial H_d}{\partial x} = h_0\sqrt{f^2 - q_x^2}, \tag{5}$$

and

$$\frac{dq_y}{ds} = -\frac{\partial H_d}{\partial y} = 0, \tag{6}$$

where $f = \sqrt{(1+\delta)^2 - q_y^2}$ is a constant during the drift. Equation (5) is solved by

$$q_x(s) = f\sin(h_0 s + \phi), \tag{7}$$

where $\phi = \sin^{-1} q_x(0)/f$.

For the coordinates, we have

$$\frac{dx}{ds} = \frac{\partial H_d}{\partial q_x} = \frac{q_x(1+h_0x)}{\sqrt{f^2 - q_x^2}}, \tag{8}$$

and

$$\frac{dy}{ds} = \frac{\partial H_d}{\partial q_y} = \frac{q_y(1+h_0x)}{\sqrt{f^2-q_x^2}}. \tag{9}$$

Using Equation (7) we can rewrite Equation 8 as

$$\int_{x(0)}^{x(s)} \frac{dx}{1+h_0x} = \int_0^s \tan(h_0 s+\phi), \tag{10}$$

giving

$$x(s) = \frac{1}{h_0}\left(-1+(1+h_0x(0))\frac{\cos\phi}{\cos(h_0s+\phi)}\right) \tag{11}$$

Using this, we can integrate Equation (9), obtaining

$$y(s) = y(0) + \frac{1+h_0x(0)}{fh_0} q_y \cos\phi\left(\tan(h_0s+\phi)-\tan\phi\right). \tag{12}$$

For the kicks, we refer to the H_f part of the Hamiltonian, giving

$$\frac{dq_x}{ds} = \frac{e}{p_0}\frac{\partial A_s}{\partial x} = -\frac{eB_y}{p_0} \tag{13}$$

and

$$\frac{dq_y}{ds} = \frac{e}{p_0}\frac{\partial A_s}{\partial y} = \frac{eB_x}{p_0}. \tag{14}$$

Derivation of the expressions for B_x and B_y is beyond the scope of this note, but uses the recursion technique [4]. In the case of a bending magnet, this is limited to 10^{th} order in x and y.

The kicks are imparted per

$$\Delta q_x = -\Delta s\frac{B_y}{R}(1+h_0x) \tag{15}$$

and

$$\Delta q_y = \Delta s\frac{B_x}{R}(1+h_0x), \tag{16}$$

where R is the beam rigidity and the $(1+h_0x)$ factors include the lengthening of the interval due to a beam offset. One might argue that this factor should be $\sqrt{1+x'^2+y'^2}+h_0x$, but then we'd have a kick depending on the momenta, which is not symplectic.

These equations are used to integrate motion in `elegant`'s `CSBEND` and `CSRCSBEND` elements.

4 Integration for a Multipole

In the case where $h_0 = 0$, the expressions are considerably simplified. In particular, both q_x and q_y are constant in the drift portion. Hence,

$$x(s) = x(0) + \frac{q_x s}{\sqrt{f^2-q_x^2}} \tag{17}$$

and

$$y(s) = y(0) + \frac{q_y s}{\sqrt{f^2-q_x^2}}. \tag{18}$$

The expressions for the kicks are given by Equations (15) and (16) with $h_0 = 0$.

These equations are used to integrate motion in `elegant`'s `KQUAD`, `KSEXT`, `KQUSE`, `FMULT`, and `MULT` elements.

5 Synchrotron Radiation

Synchrotron radiation is implemented in a less rigorous fashion. In particular, radiation kicks are imparted after the magnetic field kicks are imparted. This is discussed elsewhere [5, 6].

6 Discussion and Conclusion

We have exhibited expressions showing how `elegant` performs symplectic integration of hard-edge dipoles and multipoles. The expressions involve no approximations to the Hamiltonian or equations of motion, beyond the drift-kick-drift factorization. Hence, they are good for arbitrary momentum offset and coordinate deviations.

7 Revision Notes

Revision 1 Fixed two mistakes in transcribing equations 1 and 4. Thanks to Y. Roblin (JLab) for pointing this out.

Revision 2 Fixed some typos pointed out by L. Emery.

References

[1] M. Borland. elegant: A Flexible SDDS-Compliant Code for Accelerator Simulation. Technical Report LS-287, Advanced Photon Source, September 2000.

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[3] E. Forest. Canonical integrators as tracking codes. In M. Month and M. Dienes, editors, *AIP Conference Proceedings 184*, volume 1, pages 1106–1135, 1989.

[4] L. C. Teng. Expanded Form of Magnetic Field with Median Plane. Technical Report LCT-28, ANL, 1962.

[5] M. Borland. Does `elegant` Compute Emittance Correctly for APS? Technical Report OAG-TN-2005-021, ANL, August 2005.

[6] M. Borland. Impact of Radiation Opening Angle on ERL Emittance. Technical Report OAG-TN-2006-043, Advanced Photon Source, October 2006.