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Minimizing “R_{56}” for a non-ultrarelativistic electron beam

Elegant defines R_{56} as

$$R_{56} \equiv \frac{d\delta}{ds} \tag{1}$$

where $s \equiv \beta ct$ is pathlength and δ the fractional momentum offset. At relativistic energies, when $s = ct$, one can apply the assimilate $R_{56} = c\partial t_{final}/\partial\delta_{initial}$. At low energies the latter mapping breaks as velocity spread can be significant and needs to be accounted for. Therefore, following M. Borland's suggestions let's introduce

$$R_{76} \equiv \frac{d\delta}{dt} = \frac{d\delta}{ds}\frac{ds}{dt}, \tag{2}$$

where $ds = c(\beta dt + td\beta)$ so that

$$\begin{aligned} R_{76} &\equiv \frac{d\delta}{dt} = R_{56}\left(\beta c + ct\frac{d\beta}{dt}\right) && (3)\\ &= R_{56}\left(\beta c + ct\frac{d\beta}{d\delta}\frac{d\delta}{dt}\right) && (4)\\ &= R_{56}\left(\beta c + ctR_{76}\frac{d\beta}{d\delta}\right) && (5)\\ &= R_{56}\left(\beta c + \beta ctR_{76}\frac{1}{\beta}\frac{d\beta}{d\delta}\right) && (6)\\ &= R_{56}\left(\beta c + sR_{76}\frac{1}{\beta}\frac{d\beta}{d\delta}\right) && (7) \end{aligned}$$

Since $\delta \equiv (1/\gamma)\Delta\gamma$, $d\delta = (1/\gamma)d\gamma$. From $\beta^2 = 1 - 1/\gamma^2$ I get $d\beta = 1/(\beta\gamma^2)d\gamma/\gamma = 1/(\beta\gamma^2)d\delta$. So that

$$\frac{1}{\beta}\frac{d\beta}{d\delta} = \frac{1}{\beta^2\gamma^2}. \tag{8}$$

Inserting in Eq. (4) and rearraging I get

$$R_{76} = \frac{\beta c R_{56}}{1 - R_{56}\frac{s}{\beta^2\gamma^2}}. \tag{9}$$