Notes on the sign of the summation of the second order rdts for sextupoles

oblanco at ALBA CELLS

October 18, 2024

The goal of these short notes is to verify the sign of the summation in Eq. (46) in Chun-Xi Wang, *Explicit formulas for 2nd-order driving terms due to sextupoles and chromatic effects of quadrupoles*, ANL/APS/LS-330. March 10, 2012.

$$\sum f(i,j) \equiv \sum_{j>i} [f(i,j - f(j,i)] = \left(\sum_{j>i} - \sum_{i>j} \right) f(i,j) \tag{1}$$

The verification will be done in three steps.

- First, we confirm the notation and consistency of the signs in the original paper by J. Bengtsson *The Sextupole Scheme for the Swiss Light Source (SLS): An Analytic Approach.* SLS Note 9/97. Paul Scherrer Institute. Villigen, Switzerland. March 7, 1997.
- Second, we derive one of Wang's Equations (8-24), in particular h_{31000} because it has the simplest form.
- Third, we expand the sumation in Eq. (1) explicitly.

1 First step

We start by Bengtsson's Eq. (47)

$$\dot{\vec{\boldsymbol{x}}} = -\left(\frac{\partial H}{\partial \vec{\boldsymbol{x}}} J \right)^T \tag{2}$$

Writing only explicitely the equations on the horizontal plane we get

$$\begin{bmatrix} \dot{x} \\ \dot{p_x} \end{bmatrix} = -\left(\left[\frac{\partial H}{\partial x}, \frac{\partial H}{\partial p_x} \right] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^T = -\left[-\frac{\partial H}{\partial p_x}, \frac{\partial H}{\partial x} \right]^T = \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ -\frac{\partial H}{\partial x} \end{bmatrix} \tag{3}$$

Equation (3) is consistent with the Hamilton equations written in Eq. (19) by Bengtsson, and cited also elsewhere, for example Eq. (2.11) in Section 1, Hamiltonian for Particle Motion in Accelerators by S.Y.Lee *Accelerator Physics.* Fourth Edition. World Scientific.

This explicit form was done to confirm that the vector components are in the order $\vec{\boldsymbol{x}} = (x, p_x, \cdots)^T$.

Now we take the definition in Eq. (56)

$$:f(\vec{\boldsymbol{x}}):g(\vec{\boldsymbol{x}}) \equiv [f(\vec{\boldsymbol{x}}), g(\vec{\boldsymbol{x}})] \tag{4}$$

which is the Poisson bracket defined in Eq. (49), which is consistent with the vector components we just obtained and also consistent with the Poisson bracket definition elsewhere, for example Eq. (A.8) in Section Poisson Bracket of S.Y.Lee.

The equivalence between the Poisson bracket and the mapping of a particle trajectory through the accelerator is used in Bengtsson's Eq. (71) from where we extract only

$$M_{\vec{\boldsymbol{\xi}}0\to 1} = A^{-1} e^{\hat{V}_1} e^{\hat{V}_2} \cdots \tag{5}$$

The product $e^{\hat{V}_1} e^{\hat{V}_2}$ indicates two different elements being 1 upstream of 2. The reverse operation of matrices is flipped to a normally increasing sequence of indexes when working with maps, as noted by Bengtsson between Eqs (69) and Eq. (70).

Bengtsson's Eq (75) will define the crossed operation of these two elements

$$e^a e^b = e^{a+b+[a,b]/2} + \cdots . \tag{6}$$

This is refered as the BCH formula from quantum mechanics and signs could be verified elsewhere, for example, Eq (5.3) in B.C Hall, *Lie Groups, Lie Algebras, and representations*, Second Edition, Springer.

From this definition we jump to Bengtsson's Eq. (76) and Eq. (72),

$$\frac{1}{2}\sum_{i<j}^{N} [\hat{V}_i, \hat{V}_j]; \qquad \hat{V}_i = R_{0\to i} A_i V_i \tag{7}$$

$R_{0\to i}$ represents the phase advance from 0 to i, A_i the amplitude mapping of the beta functions at i, and V_i the magnet potential.

This Poisson bracket could be also calculated in the base (ϕ, J). We use the result Alice Quillen, *PHY411Lectures notes– Canonical Transformations*, University of Rochester, Sept. 27/2023, page 4, Section 1.2
(https://astro.pas.rochester.edu/ aquillen/phy411/lecture2.pdf).

For $f(\phi_x(x,p_x), J_x(x,p_x))$ and $g(\phi_x(x,p_x), J_x(x,p_x))$

$$[f,g] = \frac{\partial f}{\partial \phi_x}\frac{\partial g}{\partial J_x}[\phi_x, J_x] - \frac{\partial f}{\partial J_x}\frac{\partial g}{\partial \phi_x}[\phi_x, J_x] \tag{8}$$

$$+ \frac{\partial f}{\partial \phi_x}\frac{\partial g}{\partial \phi_x}[\phi_x, \phi_x] + \frac{\partial J_x}{\partial J_x}\frac{\partial g}{\partial \phi_x}[J_x, J_x] \tag{9}$$

$$\tag{10}$$

taking into account that $[\phi_x, J_x] = 1$, $[J_x, J_x] = 0$, $[\phi_x, \phi_x] = 0$ and $[\phi_x, J_x] = -[J_x, \phi_x]$ in the transformation defined in Bengtsson Eq. (89) (see here AppendixD), we get

$$[f,g]_{(x,p_x)} = [f,g]_{(\phi,J)}[\phi, J] = [f,g]_{(J,\phi)}[J,\phi] \tag{11}$$

Note that every magnet component (b_i, a_i) in Bengtsson differ in a minus sign with respect to the typical definition. Compare for example Bengtsson Eq (8) and Eq (10) with S.Y.Lee Eq (2.18) and Eq (2.19). This difference in sign swaps the sign in the resonant terms as they are written in Bengtsson's article wrt to what is implemented in tracking codes, but, it has no effect on the crossed sextupolar terms we are interested in.

With all this we have confirm the validity and the signs of the equations leading to the calculation of the second order terms.

2 Second step

We now derive one of the second order resonant terms in Wang's article.

We first confirm that the term h_{31000} has a single contributing term in the summation of all crossed terms. To do so, we use the Eq (54) in Chun-xi Wang and Alex Chao *Notes on Lie algebraic analysis of achromats.* SLAC/AP-100. Jan, 1995.

$$\begin{aligned}
&[\,|a_1b_1c_1d_1e_1\rangle\,,\,|a_2b_2c_2d_2e_2\rangle] = \\
&i(a_1b_2-a_2b_1)\,|a_1+a_2-1,b_1+b_2-1,c_1+c_2,d_1+d_2,e_1+e_2\rangle \\
&i(c_1d_2-c_2d_1)\,|a_1+a_2,b_1+b_2,c_1+c_2-1,d_1+d_2-1,e_1+e_2\rangle
\end{aligned} \tag{12}$$

Runing a small python program over the First Order Geometrical Terms in Bengtsson's article Section 5.1.2, yields

```
h31000 : i(0)|[2 1 0 0 0]>,|[2 1 0 0 0]>
h31000 : i(0)|[2 1 0 0 0]>,|[1 0 1 1 0]>
h31000 : i(-6)|[1 2 0 0 0]>,|[3 0 0 0 0]>
h31000 : i(6)|[3 0 0 0 0]>,|[1 2 0 0 0]>
h31000 : i(0)|[3 0 0 0 0]>,|[0 1 1 1 0]>
h31000 : i(0)|[1 0 1 1 0]>,|[2 1 0 0 0]>
h31000 : i(0)|[0 1 1 1 0]>,|[3 0 0 0 0]>
```

There are only two non-zero elements, h_{30000} with h_{12000} and the same in reverse order. This is the origin of the compact expression having one term positive and the other negative in Chun-xi's summation.

By looking at Chun-xi's Eq (9) for h_{31000} and Bengtsson Eq (97) we note from the phase terms that the expression was calculated using h_{30000} for k i.e. the upstream sextupole, and $(h^*_{12000})^* = h_{12000}$ for j i.e. the downstream sextupole, where we mention the index k to avoid later any confusion with the imaginary unit i.

We now proceed to calculate the term in the BCH formula using the poisson bracket $[h_{30000}, h_{12000}]$.

We will need the derivatives of the resonant base $h^{\pm}_x$ in Bengtsson's Eq (87), and they are as as follows:

$$\frac{\partial h^{\pm}_x}{\partial \phi_x} = \pm\, i h^{\pm}_x \tag{13}$$

$$\frac{\partial h^{\pm}_x}{\partial J_x} = \frac{1}{2}\frac{\sqrt{2}}{\sqrt{J_x}}e^{\pm i\phi_x} = \frac{1}{2J_x}h^{\pm}_x \tag{14}$$

We will also need the expression for the resonant terms h_{31000}, h_{12000} and their derivaties with respect to (ϕ_x, J_x).

$$f = h_{30000} = -\frac{1}{24} b_3 L \beta_x^{3/2} e^{3i\mu_x} (h_x^+)^3 \tag{15}$$

$$g = h_{12000} = -\frac{1}{8} b_3 L \beta_x^{3/2} e^{-i\mu_x} (h_x^+)^1 (h_x^-)^2 \tag{16}$$

$$\frac{\partial f}{\partial \phi_x} = -\frac{1}{24} b_3 L \beta_x^{3/2} e^{3i\mu_x} (3i) (h_x^+)^3 \tag{17}$$

$$\frac{\partial f}{\partial J_x} = -\frac{1}{24} b_3 L \beta_x^{3/2} e^{3i\mu_x} \left(\frac{3}{2J_x} \right) (h_x^+)^3 \tag{18}$$

$$\frac{\partial g}{\partial \phi_x} = -\frac{1}{8} b_3 L \beta_x^{3/2} e^{-i\mu_x} (i - 2i) (h_x^+)^1 (h_x^-)^2 \tag{19}$$

$$\frac{\partial g}{\partial J_x} = -\frac{1}{8} b_3 L \beta_x^{3/2} e^{-i\mu_x} \left(\frac{1}{2J_x} \right) (1 + 2) (h_x^+)^1 (h_x^-)^2 \tag{20}$$

$$\tag{21}$$

Part of the expression could be further simplified as

$$\left(\frac{1}{2J_x} \right) (h_x^+)^1 (h_x^+)^2 = h_x^- \tag{22}$$

Therefore, the second order term

$$h_{31000} = \frac{1}{2} [h_{30000}, h_{12000}] \tag{23}$$

$$= \frac{1}{2} \left(\frac{\partial h_{30000}}{\partial \phi_x} \frac{\partial h_{12000}}{\partial J_x} - \frac{\partial h_{30000}}{\partial J_x} \frac{\partial h_{12000}}{\partial \phi_x} \right) \tag{24}$$

$$= \frac{1}{2} \left(\frac{i}{64} b_{3k} L_k b_{3j} L_j \beta_{x,k}^{3/2} \beta_{x,j}^{3/2} e^{3i\mu_{x,k}} e^{-i\mu_{x,j}} (3 - (-1)) (h_x^+)^3 h_x^- \right) \tag{25}$$

$$= \frac{i}{32} b_{3k} L_k b_{3j} L_j \beta_{x,k}^{3/2} \beta_{x,j}^{3/2} e^{3i\mu_{x,k}} e^{-i\mu_{x,j}} (h_x^+)^3 h_x^- \tag{26}$$

$$\tag{27}$$

which matches the sign of the expression in Wang's article.

3 Explicit summation

In the summation shown in Wang's article Eq (46) it is assumed that j represents an element downstream the accelerator with respect to i.

$$\begin{aligned}
\sum f(i,j) &\equiv \sum_{j>i} [f(i,j) - f(j,i)] && (28)\\
&= \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(i,j) - \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(j,i) && (29)\\
&= \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(i,j) - \sum_{j=1}^{N} \sum_{i=j+1}^{N} f(i,j) && (30)\\
&= \left(\sum_{i=1}^{N} \sum_{j=i+1}^{N} - \sum_{j=1}^{N} \sum_{i=j+1}^{N} \right) f(i,j) && (31)\\
&= \left(\sum_{j>i}^{N} - \sum_{i>j}^{N} \right) f(i,j) && (32)
\end{aligned}$$

A The minus sign in first order terms

Here is the explanation on the minus sign between the the equations for the first order terms wrt sextupoles in Bengtsson and the Accelerator Toolbox code.

Note that this does not change the second order terms because the product of the sextupole strength cancels the minus in the definition.

Eq. (10) in Bengtsson:

$$B_y + iB_x = -\frac{P_0}{q} \sum_{n=1}^{\infty} (b_n + ia_n)(x + iy)^{n-1} \tag{33}$$

Assuming charge q, and $B_\rho = |P_o/q|$, one could write

$$B_y + iB_x = -sgn(q) B_\rho \sum_{n=1}^{\infty} (b_n + ia_n)(x + iy)^{n-1} \tag{34}$$

with sgn the sign function. In AT:

$$B_y + iB_x = B_\rho \sum_{n=1}^{\infty} (b_n + ia_n)(x + iy)^{n-1} \tag{35}$$

which differs from the paper in a minus sign and the sign of the charge. I suppose the sign of the charge is not relevant for this calculation in AT. Elegant agrees with AT in the sign convention, but the multipole components need to be scale by a factorial, i.e. for quads $b_{2,AT} = k_{1,Elegant}/(1!)$, sextupoles $b_{3,AT} = k_{2,Elegant}/(2!)$, octupoles $b_{4,AT} = k_{3,Elegant}/(3!)$, and so on.

To conclude, there is a minus between the polynom terms used by Bengtsson and the polynom in the code.

B How to calculate a first order term

I have derived h_{00200} and it should have a negative sign in AT. Here below the explanation.

I believe Eq(94) in Bengtsson has a typo, there should be a minus in front of b_2 to be consistent with Eq(10). This typo has no effect in third order RDTs, therefore, no other changes are needed. I show part of the derivation in case you would like to check it.

$$V_i = \frac{-b_2}{2(1+\delta)}(x^2 - y^2) = \frac{-b_2}{2}(1-\delta)(x^2 - y^2) \tag{36}$$

Ignoring the terms in δ

$$V_i = \frac{-b_2}{2}x^2 + \frac{b_2}{2}y^2 \tag{37}$$

Ignoring the terms in x, and using Eq(72)

$$\hat{V}_i = R_{o\to i}A_iV_i = \frac{b_2}{2}\beta_y R_{o\to i}\left(\frac{h_y^+ + h_y^-}{2}\right)^2 \tag{38}$$

which is the equivalent to Eqs(89,90,91) for the y plane. We expand the polynom on the resonant base and write the term with only h_y^+

$$\hat{V}_i = \frac{b_2}{2}\beta_y\left(\frac{1}{4}\right)(h_y^+ + h_y^-)^2 = \frac{b_2}{2}\beta_y\frac{1}{4}((h_y^+)^2e^{i2\mu_y} + \cdots) \tag{39}$$

Integrating as in Eq(63)

$$h_{00200} = \frac{1}{8}\beta_y b_2 L e^{i2\mu_y} \tag{40}$$

Changing the sign to match the AT magnetic field definition(Bengtsson has minus in Eq(10)).

$$h_{00200,AT} = -\frac{1}{8}\beta_y b_2 L e^{i2\mu_y} \tag{41}$$

C The skew quadrupole rdt

From Bengtsson's Eq(15) $H \propto -A_s$.

In Eq(63) $M_{\xi 0\to 1} = \exp\left(-\int H ds\right)$.

For a thin lens magnet this implies $M_{\xi 0\to 1} = \exp(LA_s)$, which corresponds to the expansion terms in Eq(71).

Therefore, for the k-th element (I use k to avoid confusion with the imaginary unit i)

$$V_k = L_k A_{s,k}. \tag{42}$$

Using Eq(8) for a skew quad

$$V_k = -L_k\mathbb{R}\left(\frac{ia_2}{2}(x+iy)^2\right) = a_2xyL_k \tag{43}$$

Using the resonant base in Eq(87) and applying the rotation and amplitude transformations as in Eq(72) to find $\hat{V}_k$, we get

$$V_k = a_2L_k\sqrt{\beta_x\beta_y}\left(\frac{1}{2}h_x^+e^{i\mu_x} + \frac{1}{2}h_x^-e^{-i\mu_x}\right)\left(\frac{1}{2}h_y^+e^{i\mu_y} + \frac{1}{2}h_y^-e^{-i\mu_y}\right). \tag{44}$$

We take only the term for h_{10010}.

$$h_{10010} = \frac{1}{4} a_2 L_k \sqrt{\beta_x \beta_y} e^{i\mu_x} e^{-i\mu_y}. \tag{45}$$

Bengtsson defines the magnetic field in Eq(10) with opposite sign wrt AT. Therefore,

$$h_{10010,AT} = -\frac{1}{4} a_2 L_k \sqrt{\beta_x \beta_y} e^{i\mu_x} e^{-i\mu_y}. \tag{46}$$

D The commutator of ϕ and J

The commutator $[\phi_x, J_x]$ could be calculated in the following way. First, we use Bengtsson's Eq (89). We square and add the two equations, and separatedly we divide the two equations leaving us with

$$\begin{aligned} J_x &= \frac{x^2 + p_x^2}{2} \qquad (47) \\ \phi_x &= \operatorname{atan}\left(-\frac{p_x}{x}\right) \qquad (48) \end{aligned}$$

We now use the explicit dependence to derive the commutator with the help of Maxima software.

$$\begin{aligned} [\phi_x, J_x] &= \frac{\partial \phi_x}{\partial x} \frac{\partial J_x}{\partial p_x} - \frac{\partial \phi_x}{\partial p_x} \frac{\partial J_x}{\partial x} \qquad (49) \\ &= \frac{p_x}{x^2 + p_x^2} p_x - \left(-\frac{x}{x^2 + px_x^2}\right) x \qquad (50) \\ &= 1 \qquad (51) \end{aligned}$$