Explicit Symplectic Integrators for 3D Static Magnetic Fields and Dynamic Aperture Studies with Wigglers

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November 21, 2003

Acknowledgment:

E. Forest (KEK)

D. S. Robin (LBNL)

Acknowledgment:

J. Li, V. Litvinenko, S. Mikhailov (DFELL)
C. Steier (LBNL)
J. Laskar (BdL, Paris)

Outline

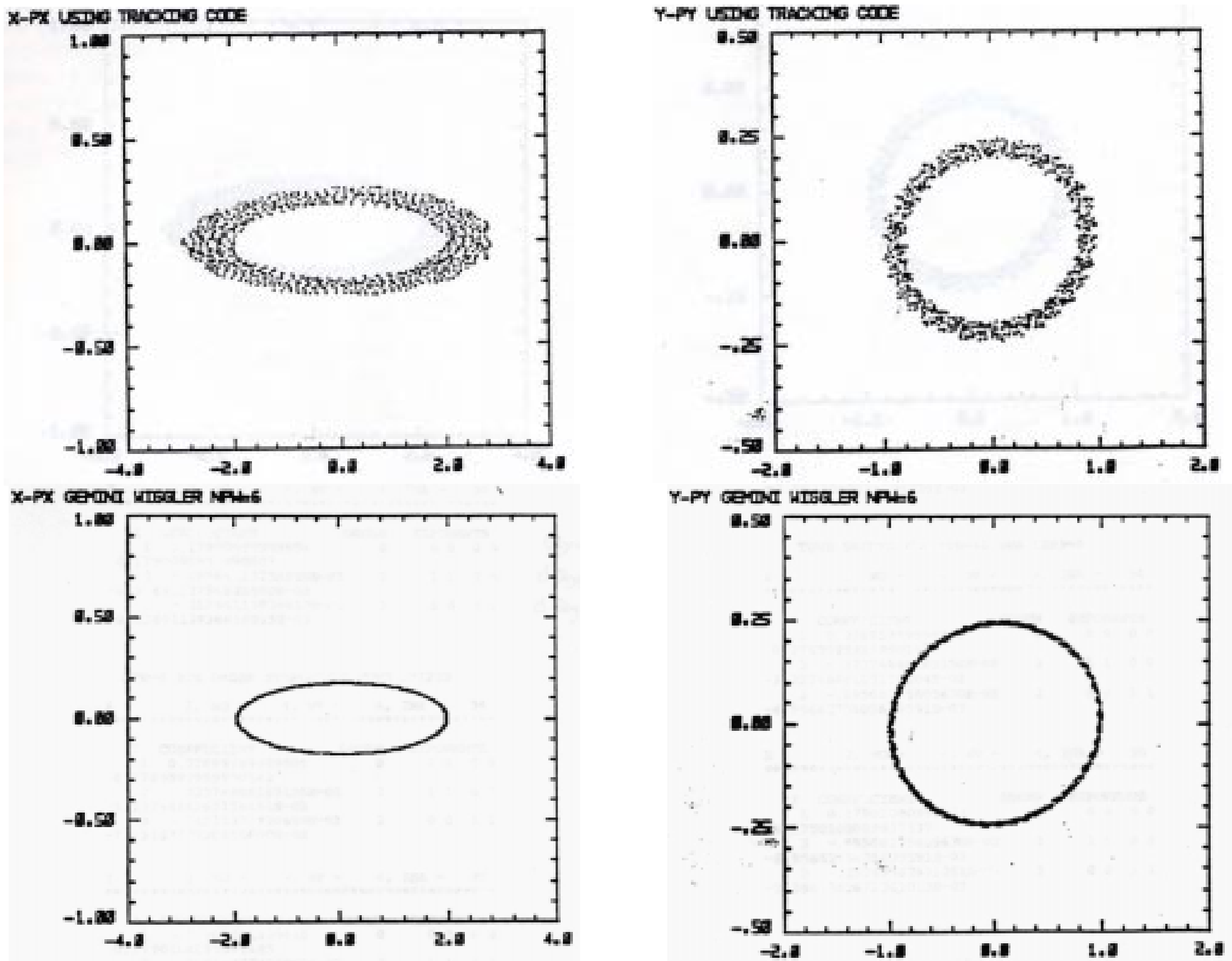
- Review: symplectic integrators for magnetic multipoles (2D fields)
- Symplectic integrators for static 3D magnetic fields
  (paraxial approximation)
- Extension to the exact Hamiltonian
- General symplectic wiggler integrator
- Duke storage ring dynamic aperture with FEL wigglers
  - Frequency map (NAFF) for dynamic aperture studies
  - Dynamics with OK4 FEL
  - Dynamics with OK5 FEL
- Summary

Symplectic vs Non-Symplectic Integration

- **Importance of symplectic integration for ring dynamics:**

  (E. Forest, ESG Tech Note-128, LBNL, 1990)

Modeling Wigglers with 2D field
(infinitely wide poles)



Non-symplectic integration

Symplectic integration

Symplectic Integration: Ruth Type Hamiltonian

- **R. Ruth Symplectic integration: Drift + Kick**

  (IEEE Trans. Nucl. Sci., ns-30, p.2669, 1983).

$$H(q,p)=T(p)+V(q)$$

- **Second-Order Lie Group Integrator**

  (E. Forest, R. Ruth, Physica D. 43, 105 (1990)

$$M(\Delta t)=\exp(:-\Delta t H(q,p):)$$

$$N_1(\frac{\Delta t}{2})=\exp(:-\frac{\Delta t}{2}T(p):),\qquad N_2(\Delta t)=\exp(:-\Delta t V(q):),$$

$$M(\Delta t)\approx N_1(\frac{\Delta t}{2})N_2(\Delta t)N_1(\frac{\Delta t}{2})+O((\Delta t)^3)$$

$$N_1\begin{pmatrix} q \\ p \end{pmatrix}=\begin{pmatrix} q+\dfrac{\partial T}{\partial p}\dfrac{\Delta t}{2} \\ p \end{pmatrix},\qquad N_2\begin{pmatrix} q \\ p \end{pmatrix}=\begin{pmatrix} q \\ p-\dfrac{\partial V}{\partial q}\Delta t \end{pmatrix}$$

Symplectic Integration for Magnetic Multipoles

Impulse Boundary Approximation

$$\boldsymbol{B}(\boldsymbol{r})=\begin{cases}\boldsymbol{B}(x,y), & inside\,boundary\\ 0, & outside\,boundary\end{cases}$$

- **Multipole Expansion**

$$\boldsymbol{A}=(0,0,A_z(x,y))$$

$$B_y+iB_x=(B_o\rho_0)\sum_{n=0}(x+iy)^n(b_n+ia_n)$$

- **Charged Particle Hamiltonian**

$$A_z(x,y)=-(B_o\rho_0)\Re\sum_{n=0}\frac{(x+iy)^{n+1}}{n+1}(b_n+ia_n)$$

$$H(x,p_x,y,p_y,\delta,l;z)=-\sqrt{(1+\delta)^2-p_x^2-p_y^2}-a_z(x,y)=H_1(p_x,p_y,\delta)+H_2(x,y)$$

$$p_{x,y}=\frac{P_{x,y}}{P_0},\qquad a_z(x,y)=\frac{qA_z(x,y)}{P_0c}$$

Review: Symplectic Integration for Wigglers/Undulators

- **Wiggler Hamiltonian with Paraxial Approximation**

$$H(x,p_x,y,p_y,\delta,l;z)=-\sqrt{(1+\delta)^2-(p_x-a_x(x,y,z))^2-(p_y-a_y(x,y,z))^2}-a_z(x,y,z)$$

$$H(x,p_x,y,p_y,\delta,l;z)\approx-\delta+\frac{(p_x-a_x(x,y,z))^2}{2(1+\delta)}+\frac{(p_y-a_y(x,y,z))^2}{2(1+\delta)}-a_z(x,y,z)$$

- **Average Hamiltonian Model**

$$\boldsymbol{a}(\boldsymbol{r})=(a_x(x,y,z),a_y(x,y,z),0)$$

$$H(x,p_x,y,p_y,\delta,l;z)\approx\left\{-\delta+\frac{p_x^2+p_y^2}{2(1+\delta)}\right\}+\left\{\frac{\langle a_x^2\rangle_{\lambda_w}+\langle a_y^2\rangle_{\lambda_w}}{2(1+\delta)}\right\}$$

An another method: averaging around the equilibrium orbit (fast orbit),
Lloyd Smith, LBNL ESG Tech Note-24, 1986.

- **Generating Function Method: expand GF in Taylor series**

  (J. Bahrdt and G. Wustefeld, BESSY TB Nr. 158 and 163, BESSY Gmbh, 1990-1991).
  - **Implicit method;**
  - **Limited orders;**
  - **Map with respect to design orbit;**

Explicit Symplectic Integrator for 3D Field

- **Hamiltonian with a 3D Magnetic Field (paraxial approximation)**

$$H(x,p_x,y,p_y,\delta,l;z)\approx -\delta+\frac{(p_x-a_x(x,y,z))^2}{2(1+\delta)}+\frac{(p_y-a_y(x,y,z))^2}{2(1+\delta)}-a_z(x,y,z)$$

- **Extending the phase space to 4D**

$$K(x,p_x,y,p_y,\delta,l,z,p_z;\sigma)\approx -\delta+\frac{(p_x-a_x)^2}{2(1+\delta)}+\frac{(p_y-a_y)^2}{2(1+\delta)}-a_z+p_z$$

$$d\sigma=dz$$

- **Lie Map "Solution" in 4D**

$$\frac{\partial K}{\partial \sigma}=0,$$

$$M(\Delta\sigma)=\exp(:-\Delta\sigma K:)$$

Explicit Symplectic Integrator for 3D Field

- **Breaking into Solvable Parts**

$$K(\sigma)=\{p_z\}-\{a_z\}+\left\{\frac{(p_y-a_y)^2}{2(1+\delta)}\right\}+\left\{-\delta+\frac{(p_x-a_x)^2}{2(1+\delta)}\right\}=H_1+H_2+H_3+H_4$$

$$N_i(\Delta\sigma)=\exp(:-\Delta\sigma H_i:)$$

- **Solving** $H_3(p_y-a_y(x,y,z))$

$$H_3=\frac{(p_y-a_y(x,y,z))^2}{2(1+\delta)}=A_y\frac{p_y^2}{2(1+\delta)},\qquad A_y=\exp(:-\int_{y_0}^{y}a_y(x,y,z)dy:)$$

$$A_y\{x,y,z,l,\delta\}=\{x,y,z,l,\delta\}\qquad A_y^{-1}\{x,y,z,l,\delta\}=\{x,y,z,l,\delta\}$$

$$A_y p_x=p_x-\int_{y_0}^{y}\frac{\partial a_y(x,y,z)}{\partial x}dy\qquad A_y^{-1}p_x=p_x+\int_{y_0}^{y}\frac{\partial a_y(x,y,z)}{\partial x}dy$$

$$A_y p_y=p_y-a_y(x,y,z)\qquad A_y^{-1}p_y=p_y+a_y(x,y,z)$$

$$A_y p_z=p_z-\int_{y_0}^{y}\frac{\partial a_y(x,y,z)}{\partial z}dy\qquad A_y^{-1}p_z=p_z+\int_{y_0}^{y}\frac{\partial a_y(x,y,z)}{\partial z}dy$$

Explicit Symplectic Integrator for 3D Field

- **Second-Order Explicit Scheme**

$$\exp(:-\Delta\sigma K:) = \exp(:-\frac{\Delta\sigma}{2} p_z:)\exp(:\frac{\Delta\sigma}{2} a_z(x,y,z):)$$

$$\left\{A_y \exp(:-\frac{\Delta\sigma}{2}\frac{p_y^2}{2(1+\delta)}:)A_y^{-1}\right\}$$

$$\left\{A_x \exp(:-\Delta\sigma(-\delta+\frac{p_x^2}{2(1+\delta)}):)A_x^{-1}\right\}$$

$$\left\{A_y \exp(:-\frac{\Delta\sigma}{2}\frac{p_y^2}{2(1+\delta)}:)A_y^{-1}\right\}$$

$$\exp(:\frac{\Delta\sigma}{2} a_z(x,y,z):)\exp(:-\frac{\Delta\sigma}{2} p_z:) + O((\Delta\sigma)^3)$$

Higher Order Symplectic Integrator

- **Iterative Method of Yoshida**

  (H. Yoshida, Phys. Lett. A, vol. 150, p 262, 1990)

  - **Start from a (2n)th- order approximated map**

$$M_{2n}(\Delta\sigma)=\exp(:-\Delta\sigma K+(\Delta\sigma)^{2n+1}F_{2n+1}+O(\Delta\sigma)^{2n+3}:)$$

  - **Construct a symmetrized map with time reversibility**

$$M_{2n+2}(\Delta\sigma)=M_{2n}(x_1\Delta\sigma)M_{2n}(x_0\Delta\sigma)M_{2n}(x_1\Delta\sigma)$$

$$\approx\exp(:-(2x_1+x_0)\Delta\sigma K+(2x_1^{2n+1}+x_0^{2n+1})(\Delta\sigma)^{2n+1}F_{2n+1}+O(\Delta\sigma)^{2n+3}:)$$

$$=\exp(:-\sigma K+O(\Delta\sigma)^{2n+3}:)$$

When

$$2x_1+x_0=1, \qquad 2x_1^{2n+1}+x_0^{2n+1}=0$$

One real solution is:

$$x_1=\frac{1}{2-2^{1/(2n+1)}}, \qquad x_0=-\frac{2^{1/(2n+1)}}{2-2^{1/(2n+1)}}$$

Time-Domain Integration

- **Quadratic Lagrangian in 4-space**

$$L(x^i, U^i, \tau) = -\frac{m}{2} U_i U^i - \frac{q}{c} U_i A^i,$$

$$x^i = (ct, x, y, z), U^i = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z)$$

$$P^i = m U^i + \frac{q}{c} A^i, \qquad H(\tau) = P^i U_i + L = \frac{(P^i - \frac{q}{c} A^i)(P_i - \frac{q}{c} A_i)}{2m}$$

- **With some manipulations (for static magnetic field):**

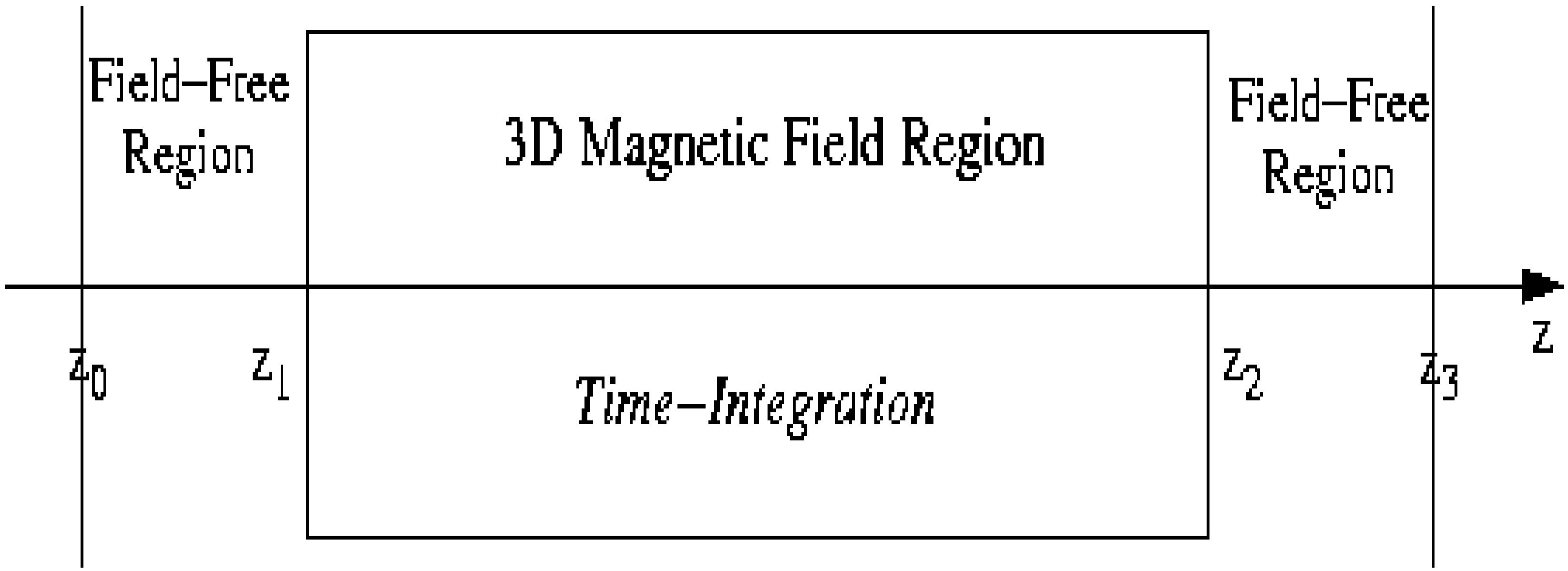
$$K(x, p_x, y, p_y, z, p_z, \delta, l; \sigma) = \frac{(p_x - a_x)^2 + (p_y - a_y)^2 + (p_z - a_z)^2 - (1+\delta)^2}{2\sqrt{(1+\delta)^2 + 1/(\gamma_0^2 \beta_0^2)}}$$

$$d\sigma = c\, dt$$

From Time-Integration to Space-Integration

- **Transition has to be performed in the field-free region**

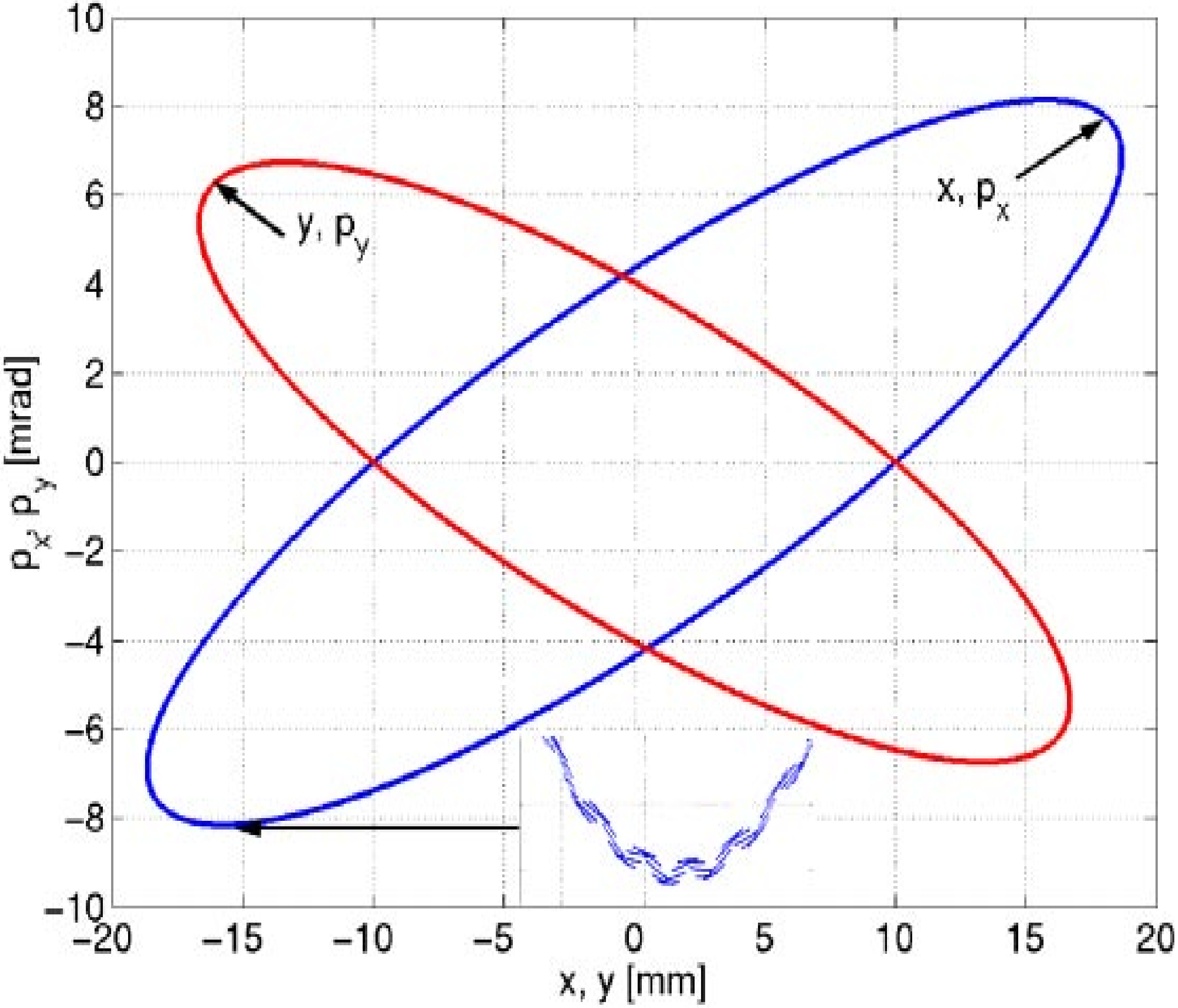
  Numerically, the field value is at the same level of truncation error



$$K_{drift}(x,p_x,y,p_y,z,p_z,\delta,l;\sigma)=-\sqrt{(1+\delta)^2-p_x^2-p_y^2}+p_z$$

FODO Lattice with Exact Hamiltonian

- **A FODO Lattice**



$$\boldsymbol{a}(\boldsymbol{r})=(0,0,-\frac{1}{2}b(z)(x^2-y^2))$$

$$b(z)=\frac{b_1}{\sqrt{2\pi}}\exp(\frac{-z^2}{2\,l_q^2})$$

General Insertion Device Symplectic Integrator

- **Horizontal Wiggler Magnetic Field**

$$B_y = -B_0 \sum_{m,n} C_{m,n} \cos(k_{xl} x) \cosh(k_{ym} y) \cos(k_{zn} + \theta_{zn})$$

$$B_x = B_0 \sum_{m,n} C_{m,n} \frac{k_{xl}}{k_{ym}} \sin(k_{xl} x) \sinh(k_{ym} y) \cos(k_{zn} + \theta_{zn})$$

$$B_z = B_0 \sum_{m,n} C_{m,n} \frac{k_{zn}}{k_{ym}} \cos(k_{xl} x) \sinh(k_{ym} y) \sin(k_{zn} + \theta_{zn})$$

$$K_w = \frac{eB}{mc^2 k_w}$$

$$k_{zn} = n k_w$$

- **Vector Potential**

$$\boldsymbol{a} = \frac{q}{P_0 c}(A_x, A_y, 0)$$

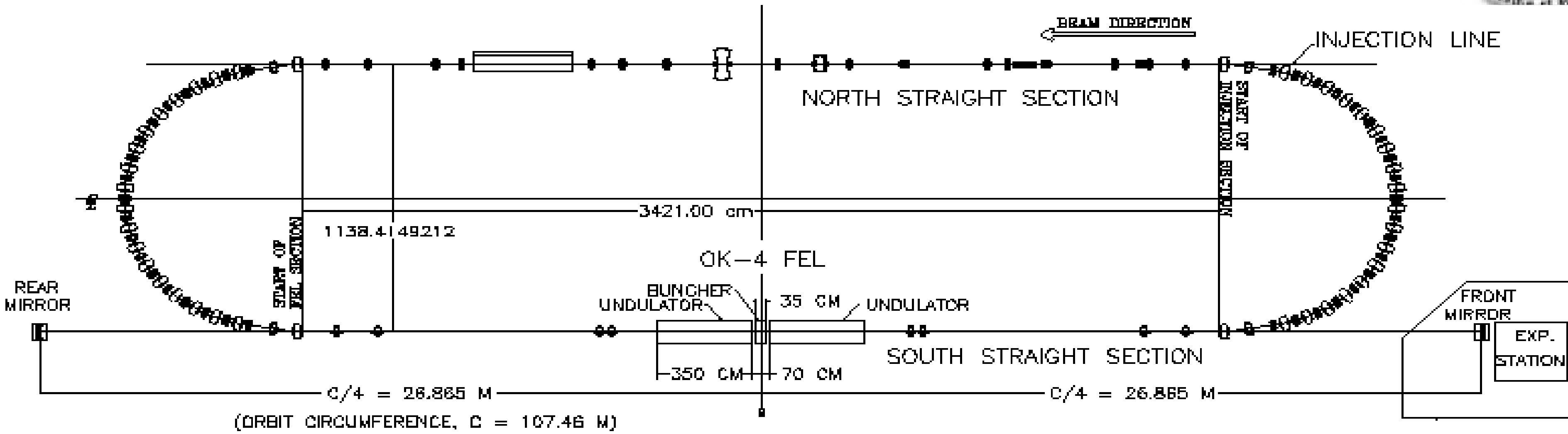
$$a_x = \frac{K_w}{\gamma_0 \beta_0} \sum_{m,n} \frac{C_{m,n}}{k_{zn}/k_w} \cos(k_{xl} x) \cosh(k_{ym} y) \sin(k_{zn} + \theta_{zn})$$

$$a_y = \frac{K_w}{\gamma_0 \beta_0} \sum_{m,n} \frac{C_{m,n}}{k_{zn}/k_w} \frac{k_{xl}}{k_{ym}} \sin(k_{xl} x) \sinh(k_{ym} y) \sin(k_{zn} + \theta_{zn})$$

- **Arbitrarily Polarized Wiggler**

$$\boldsymbol{B} = \boldsymbol{B}_{\text{hori wiggler}} + \boldsymbol{B}_{\text{vert wiggler}} \qquad \boldsymbol{a} = \boldsymbol{a}_{\text{hori wiggler}} + \boldsymbol{a}_{\text{vert wiggler}}$$

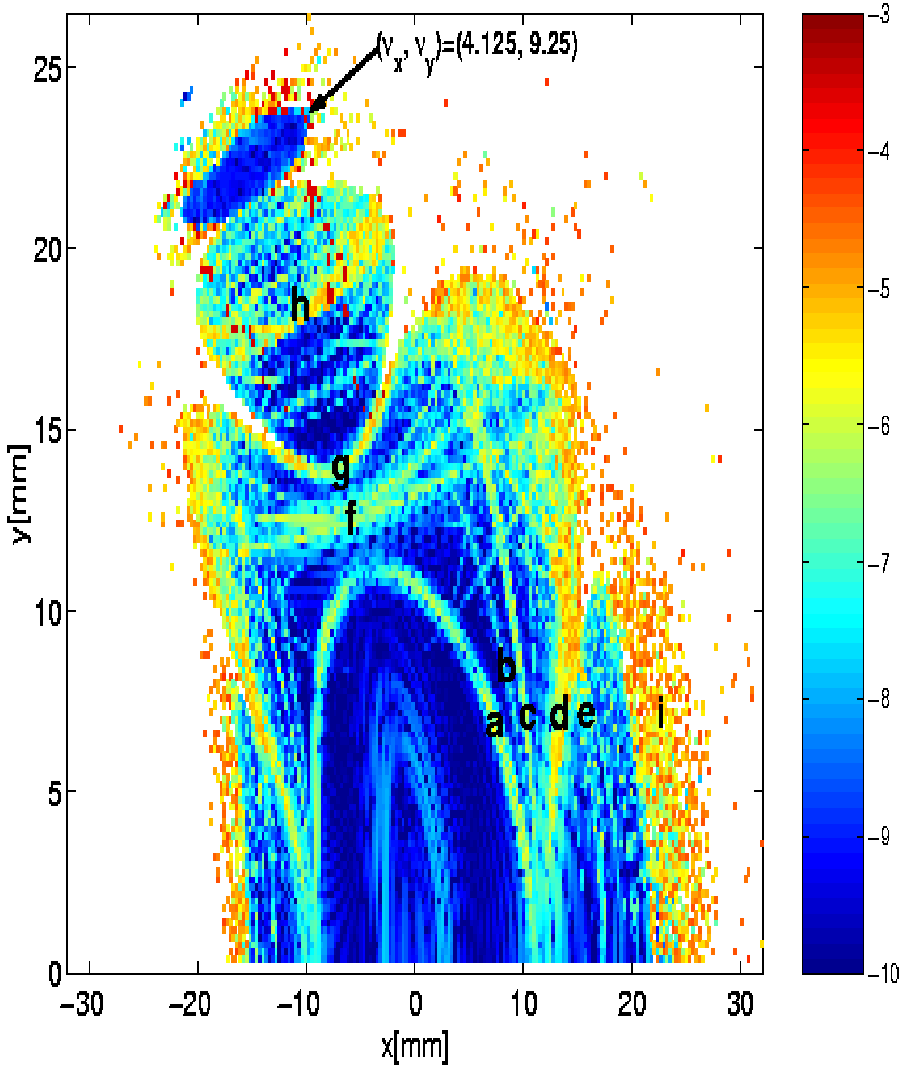
Duke FEL Storage Ring with OK4 FEL

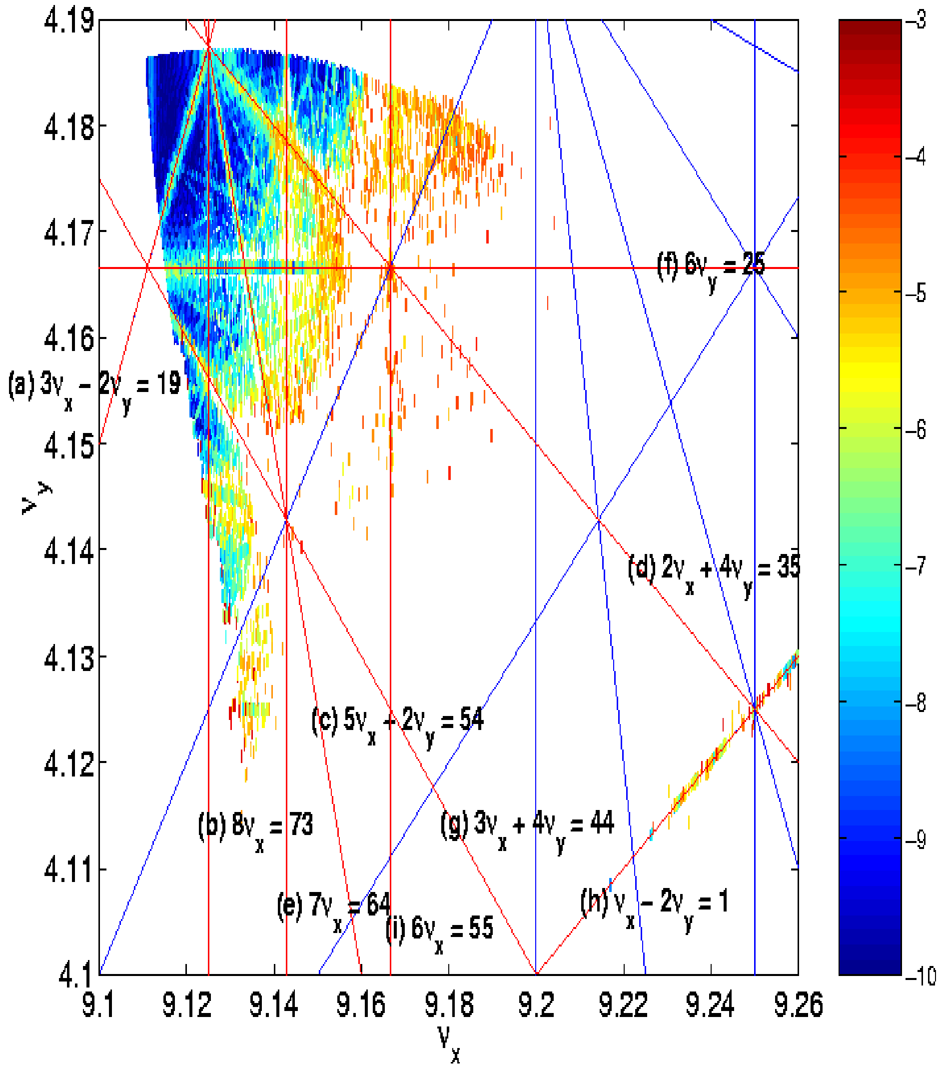


- **Duke Storage ring**
  - **Energy 0.25 to 1.2 GeV**
  - **Circumference 107 m**
  - **Emittance 18 nm (1 GeV)**
- **OK4 FEL (since 1996)**
  - **Two planar wigglers, L_w = 3.4 m, λ_w = 10 cm, max K_w= 5.1**
  - **Wavelength: 193.7 to 2100 nm**
- **Gamma-ray Source**
  - **Linearly polarized**
  - **Energy: 0.7 to 58 MeV**
  - **Energy spread (FWHM): 0.4 - 2%**

Transverse Dynamics: Bare Lattice, β_x=2.48m, β_y=1.56m, chrom = (0, 0)

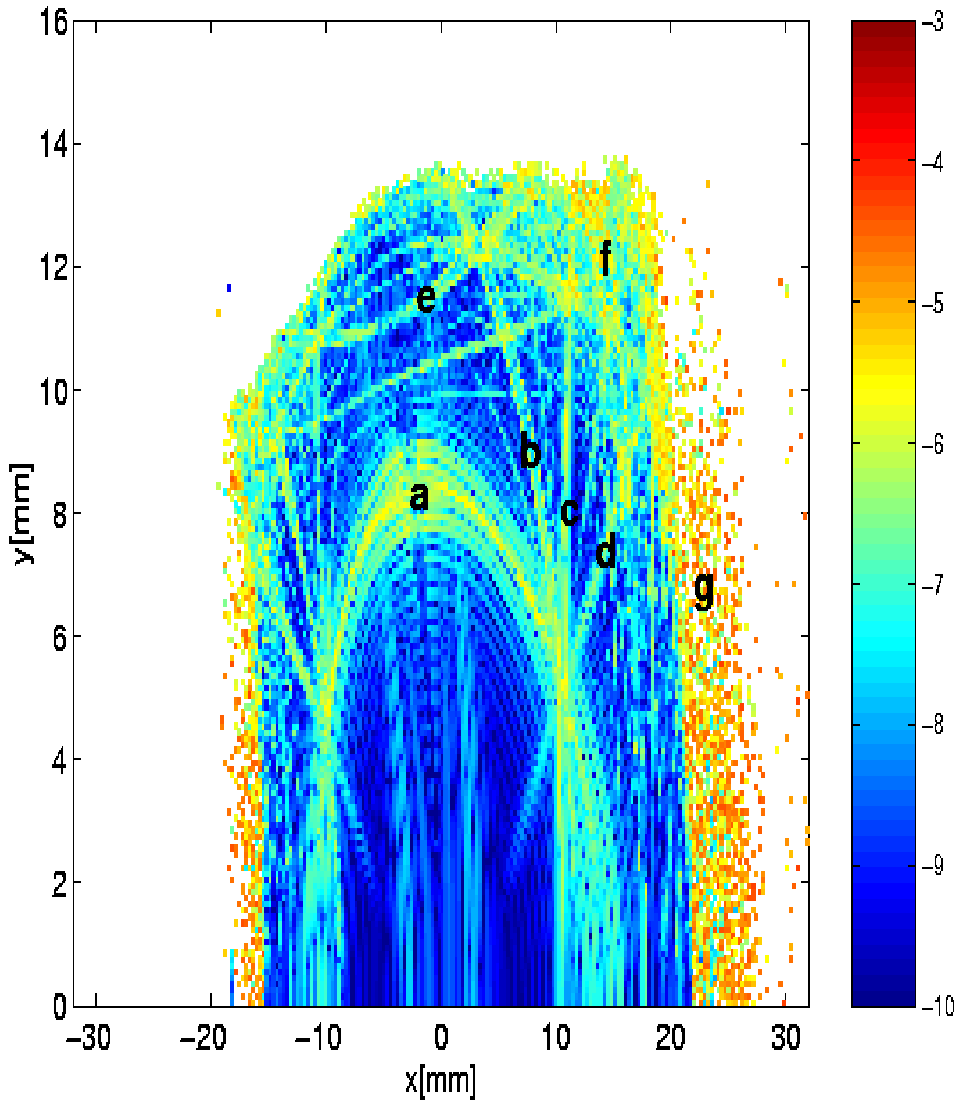
- **OK4 FEL OFF**

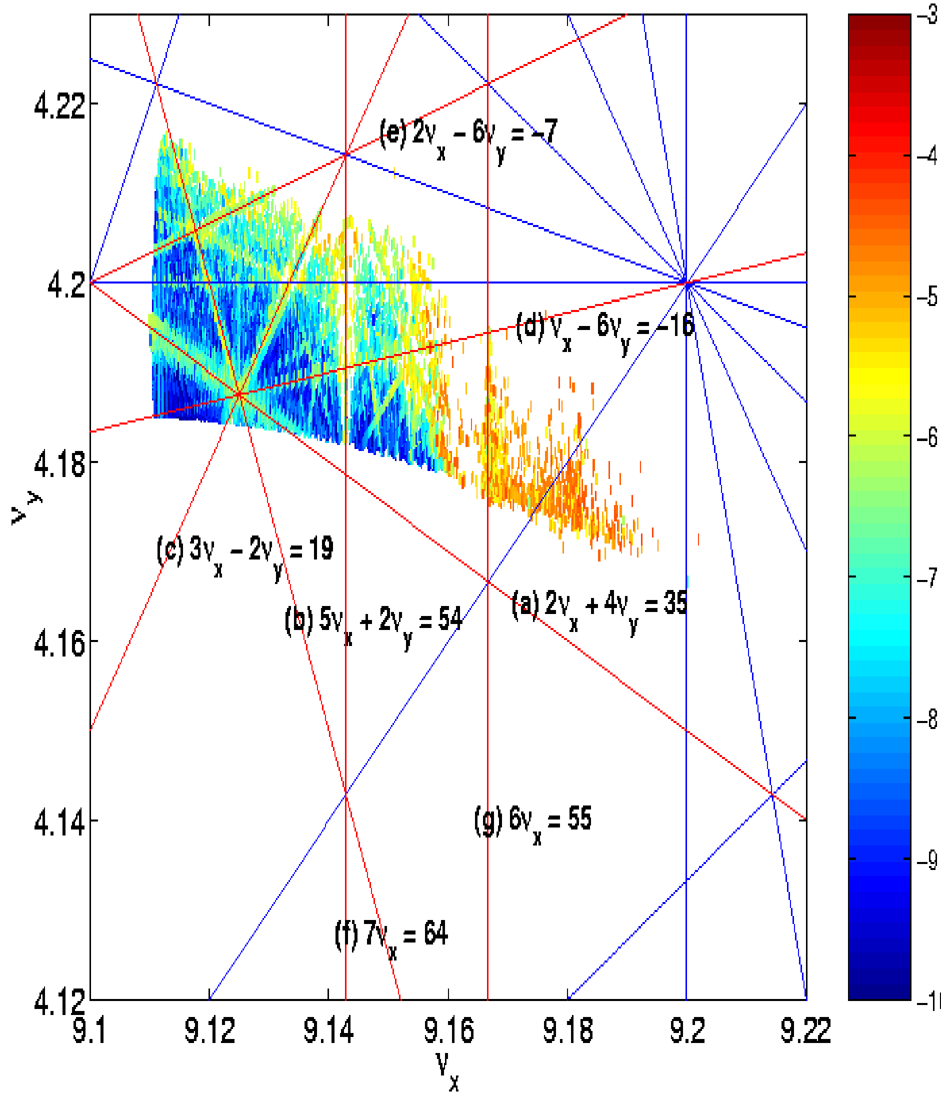




Transverse Dynamics: OK4 FEL ON, β_x=2.48m, β_y=1.56m, chrom = (0, 0)

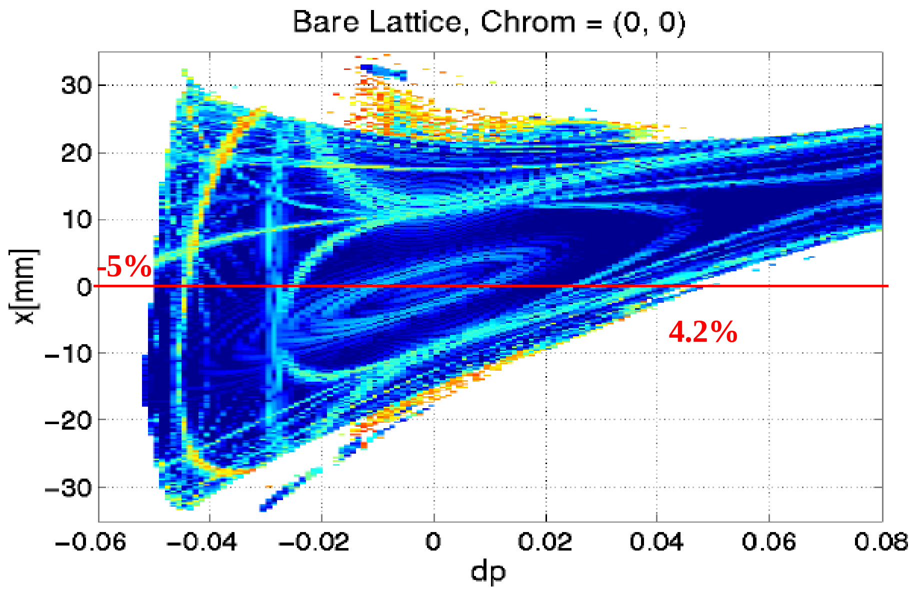
- **1 GeV Bare Lattice, OK4 ON, K_w= 5, 2nd order integrator, 5 slices per period**

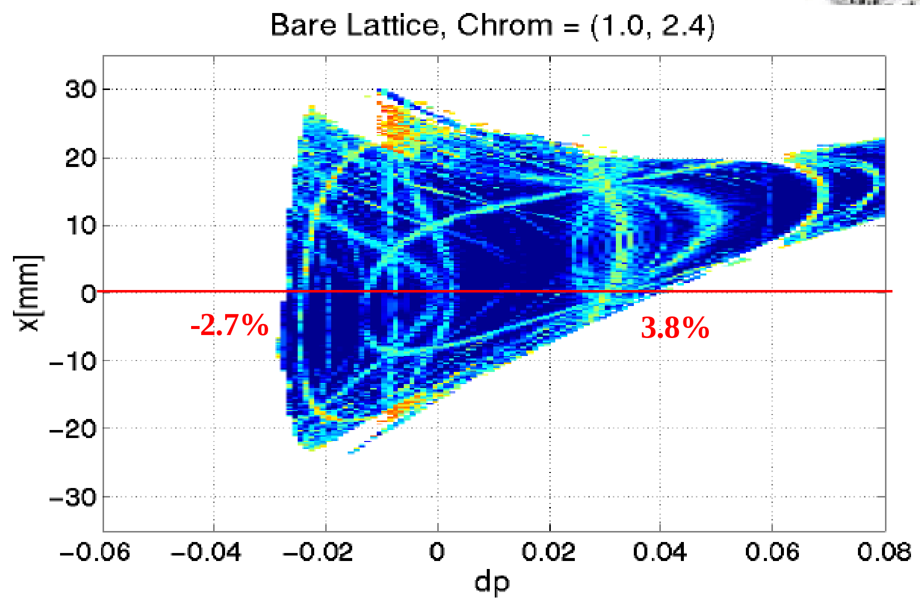


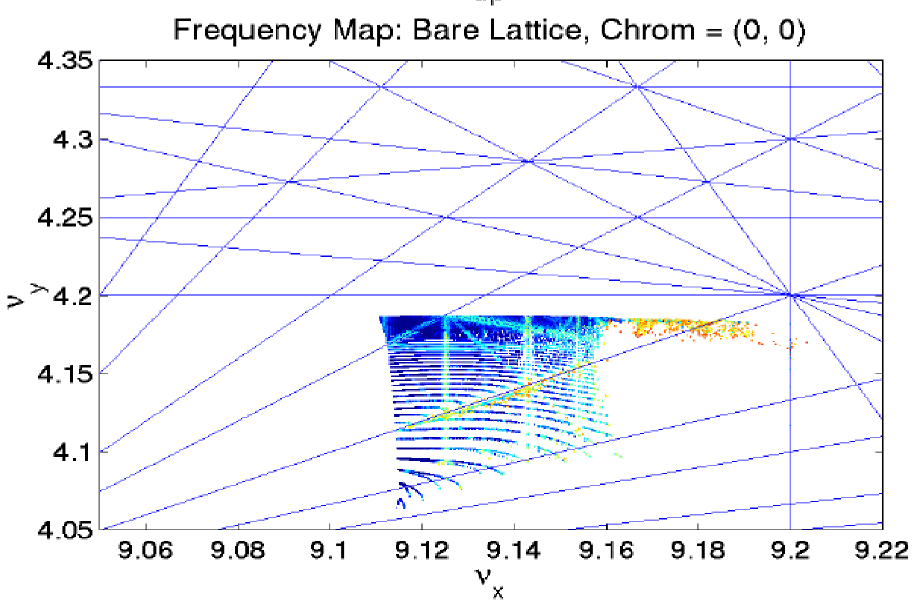


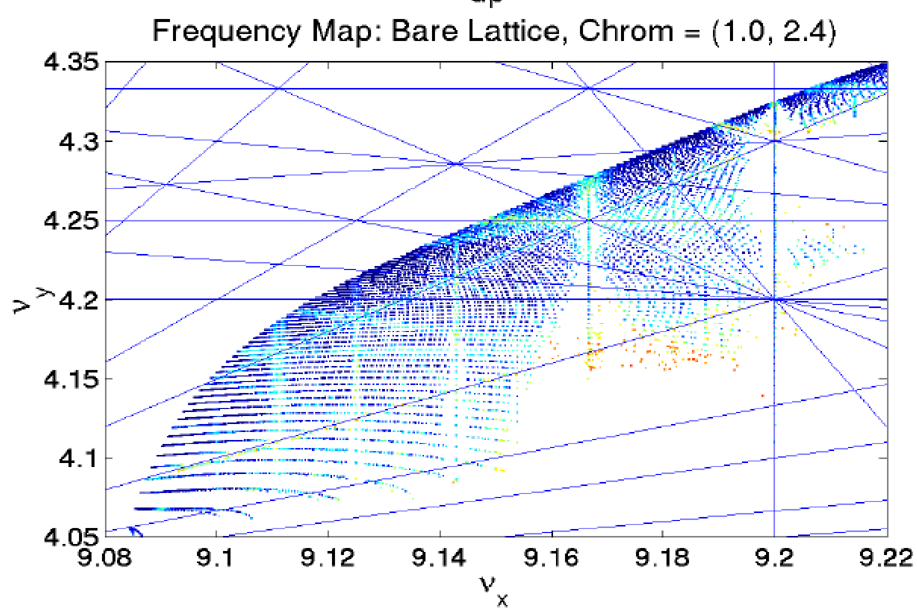
Frequency Map for Dynamics Studies

Energy Aperture: Bare Lattice (Arc Center), β_x=2.48m, β_y=1.56m

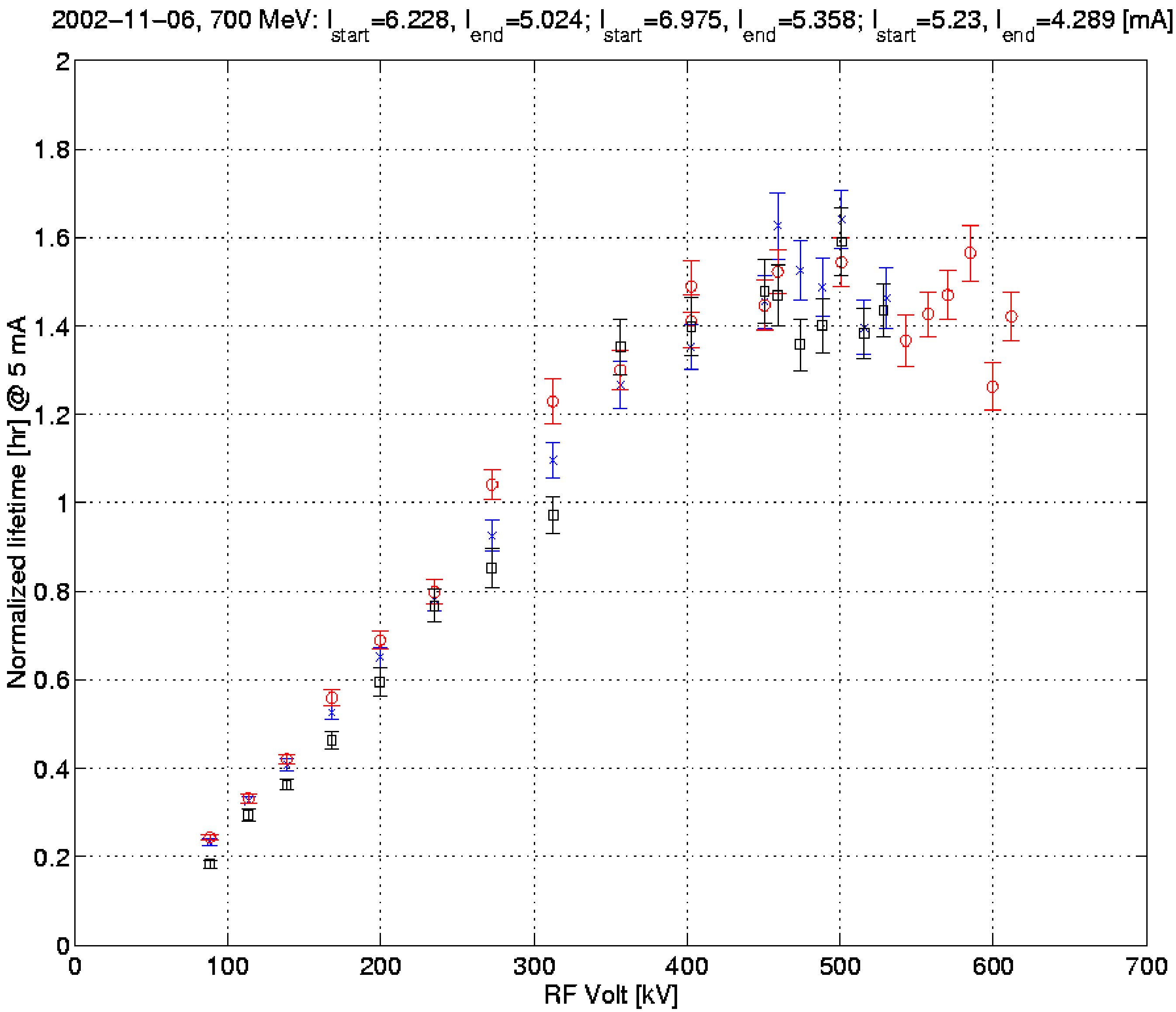








Measured Energy Aperture from RF Scan

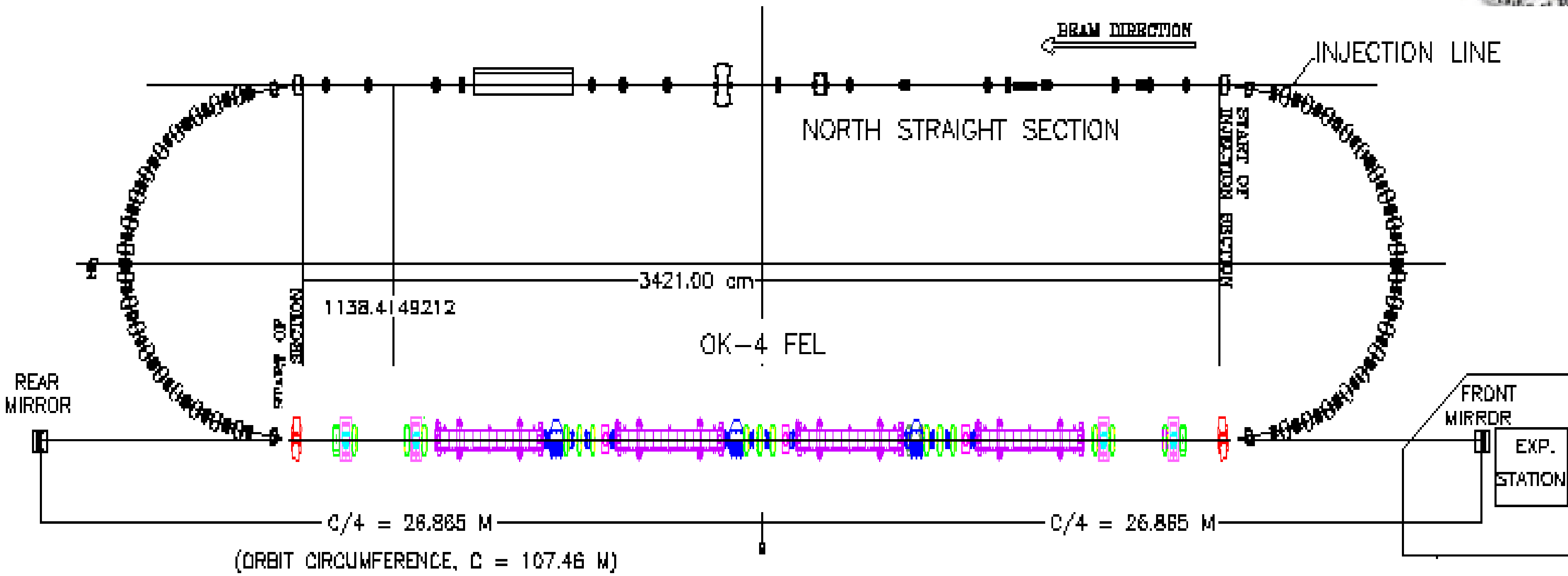


700MeV, OK4 Off

Tunes = (9.109, 4.195)
Chros = (0.97, 2.98)

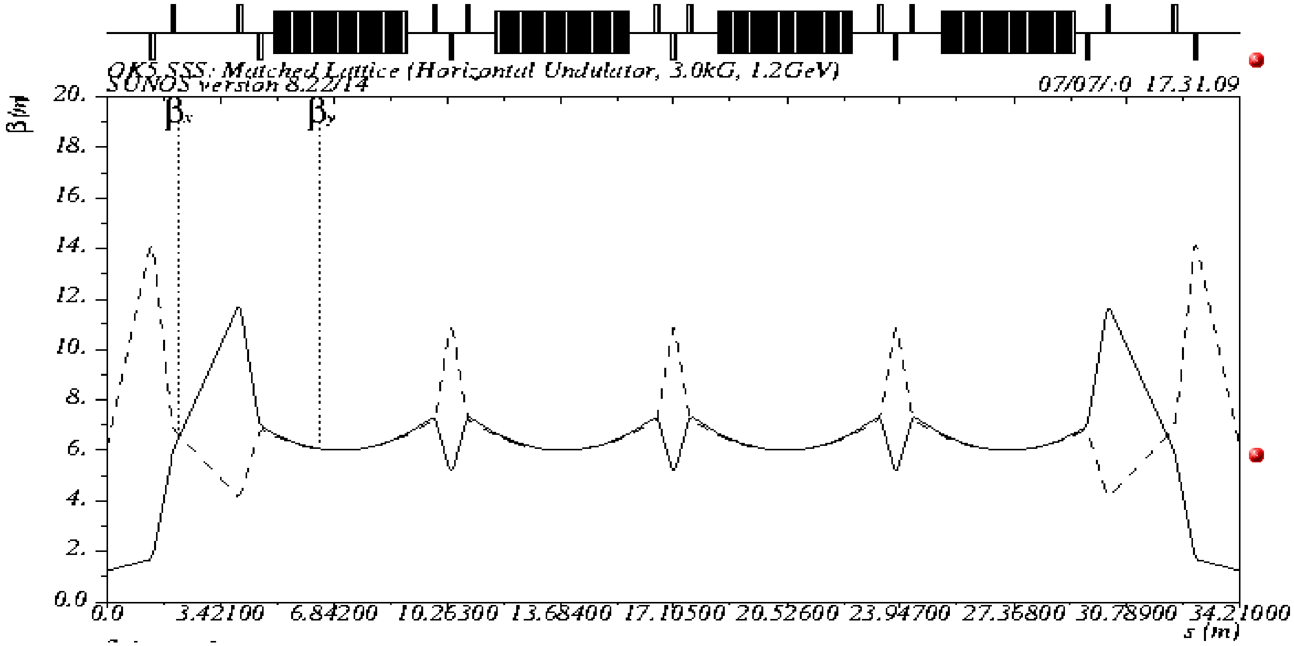
Dynamic Energy Aperture = 2.7%

OK5 FEL Project at Duke



- **OK5 FEL and Gamma-ray Source**
  - **Four variably polarized wigglers, L_w = 4.04 m, λ_w = 12 cm, max K_w= 3.3**
  - **Higher FEL gain and wider wavelength coverage**
  - **Linearly and circularly polarized gamma-ray beams**
  - **Higher gamma-ray flux: 10x - 100x;**

OK5 Wigglers and Dynamics Challenges



- **Dynamic aperture problem with BL11 wiggler at SPEAR**

  (J. Safranek, et al., Phy. Rev.-SP, v.5, 010701, 2002)

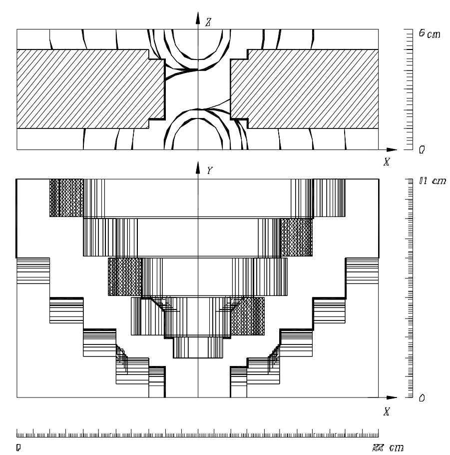
  L_w=2.3m, K_w= 33.7,

  K_w/γ = 5.6E-3 (3 GeV)

- **Duke ring with OK5 wigglers**

  L_w=16m, K_w= 2.53,

  K_w/γ = 4.3E-3 (300 MeV)



OK5 Wiggler Field Modeling

- **3D Field Simulation**
  - **Mermaid, a 2D/3D magnet design code, A. N. Dubrovin, Novosibirsk, Russia**
  - **S. Mikhailov, DFELL**
- **Field Data on a x-y Grid (128 harmonics)**

$$B_y = \sum_{n=1}^{128} f_n(x,y)\cos(n k_w z)$$

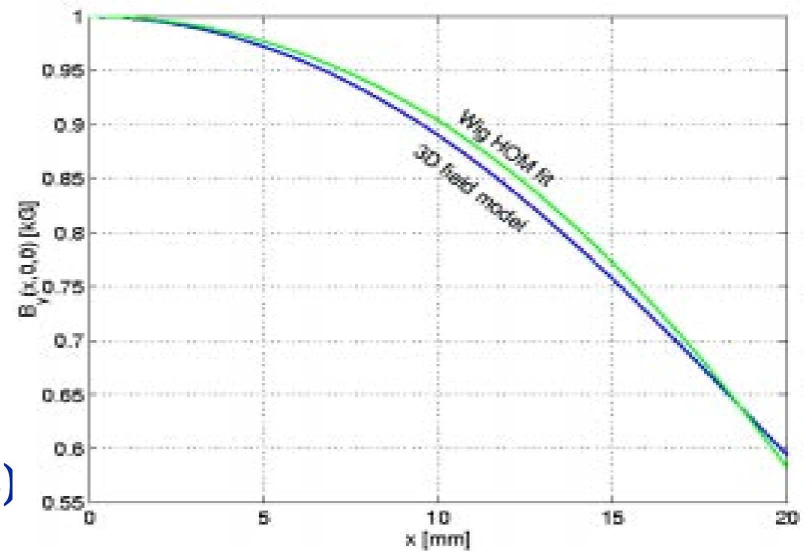
- **Simplified Field Model (plain wiggler, 3 modes)**

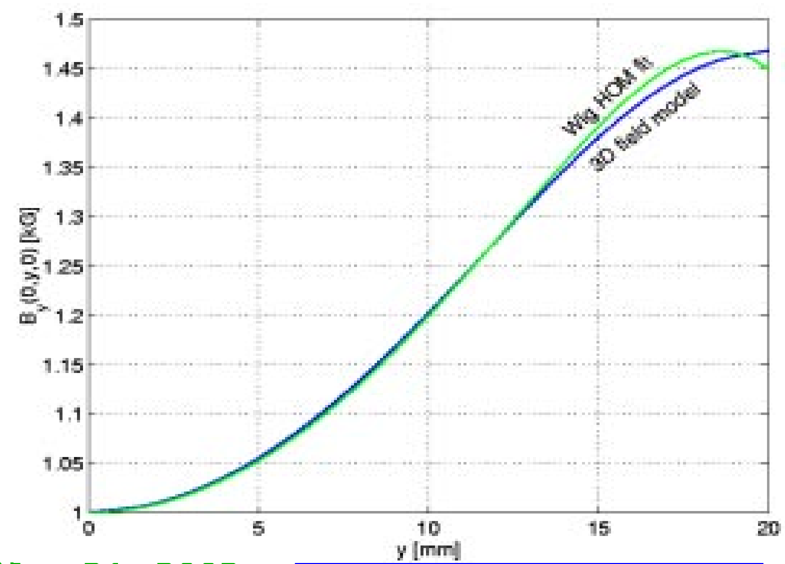
$$B_y = -B_0 \sum_{m,n} C_{m,n} \cos(k_{xl} x)\cosh(k_{ym} y)\cos(k_{zn} + \theta_{zn})$$

$$C_1 = 1.025, \frac{k_{x1}}{k_w} = 1.0592, \frac{k_{y1}}{k_w} = 1.4567, \frac{k_{z1}}{k_w} = 1$$

$$C_2 = -0.0094, \frac{k_{x2}}{k_w} = 4.1272, \frac{k_{y2}}{k_w} = 4.2466, \frac{k_{z2}}{k_w} = 1$$

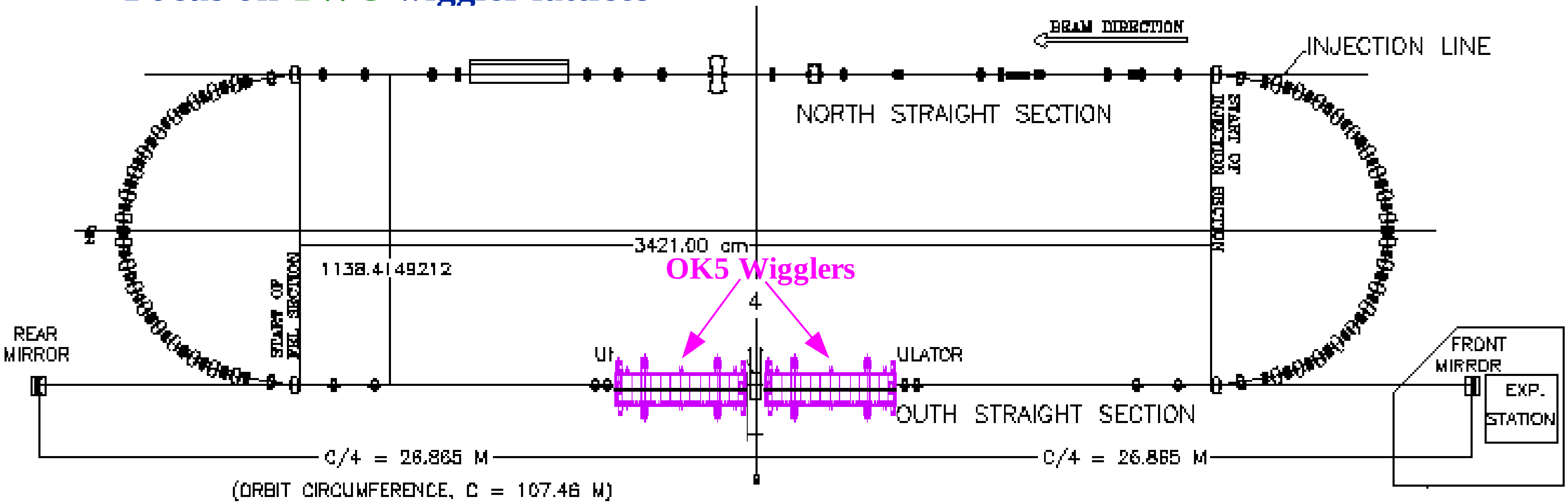
$$C_3 = -0.0156, \frac{k_{x3}}{k_w} = 2.9555, \frac{k_{y3}}{k_w} = 4.2113, \frac{k_{z3}}{k_w} = 3$$





Dynamics with OK5 Wigglers

- **Focus on TWO wiggler lattices**



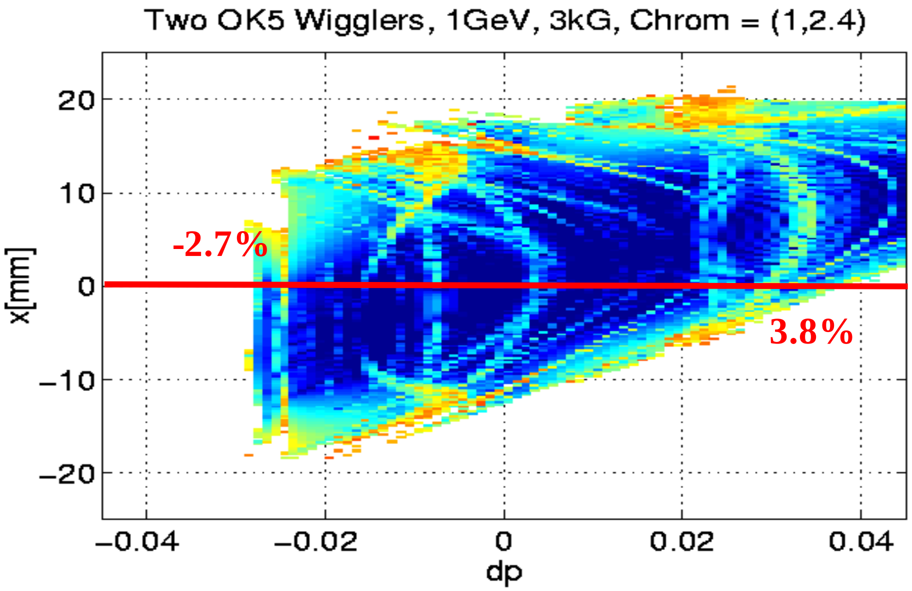
- **Three Different OK5 Wiggler Operation Modes**
  - **1 GeV, 3 kG, Circularly Polarized**

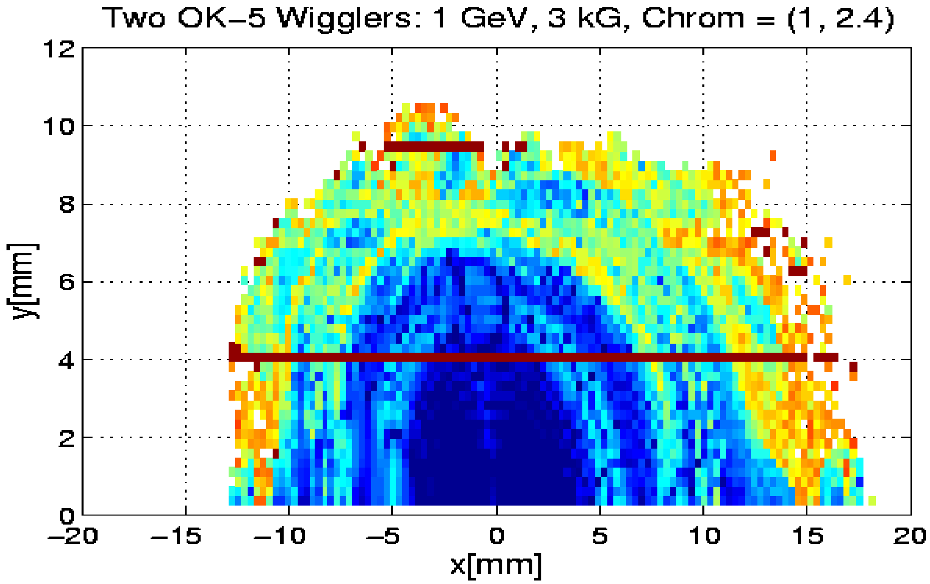
    λ = 193 nm, E_γ = 99 MeV
  - **300 MeV, 1.8 kG, Circularly Polarized**

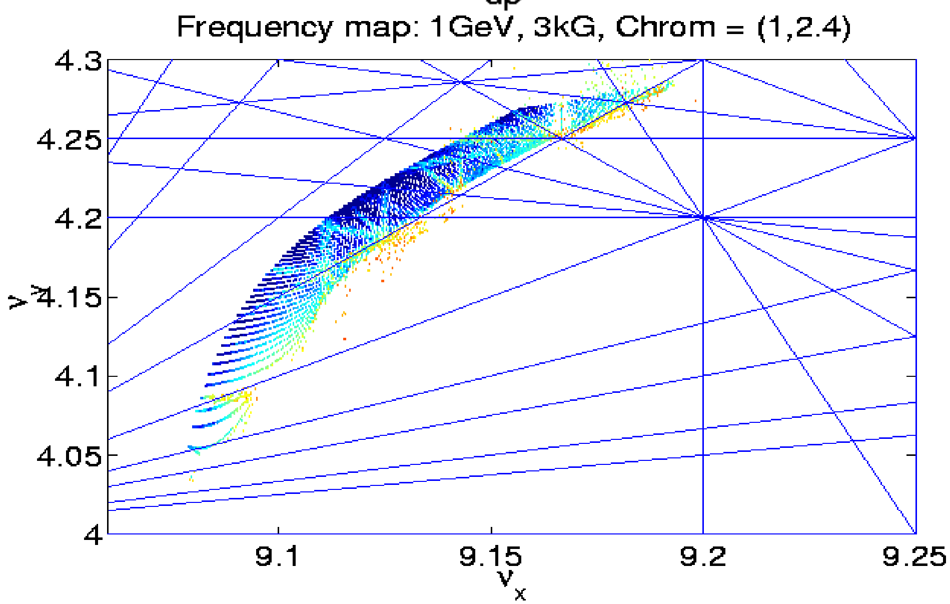
    λ = 882 nm, E_γ = 1.94 MeV
  - **300 MeV, 2.55 kG, Linearly Polarized**

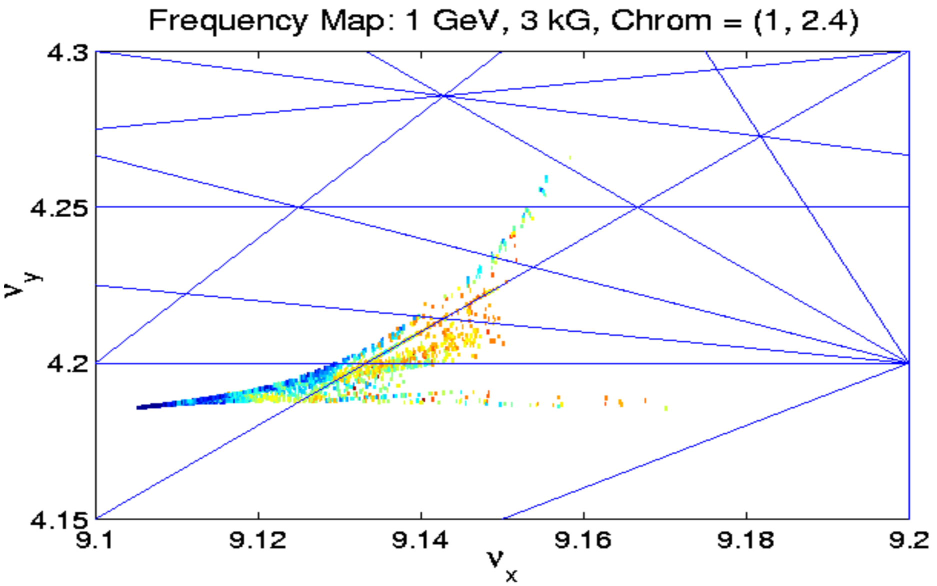
    λ = 882 nm, E_γ = 1.94 MeV

Dynamics with OK5 Wigglers: 1 GeV, 3 kG each wiggler array

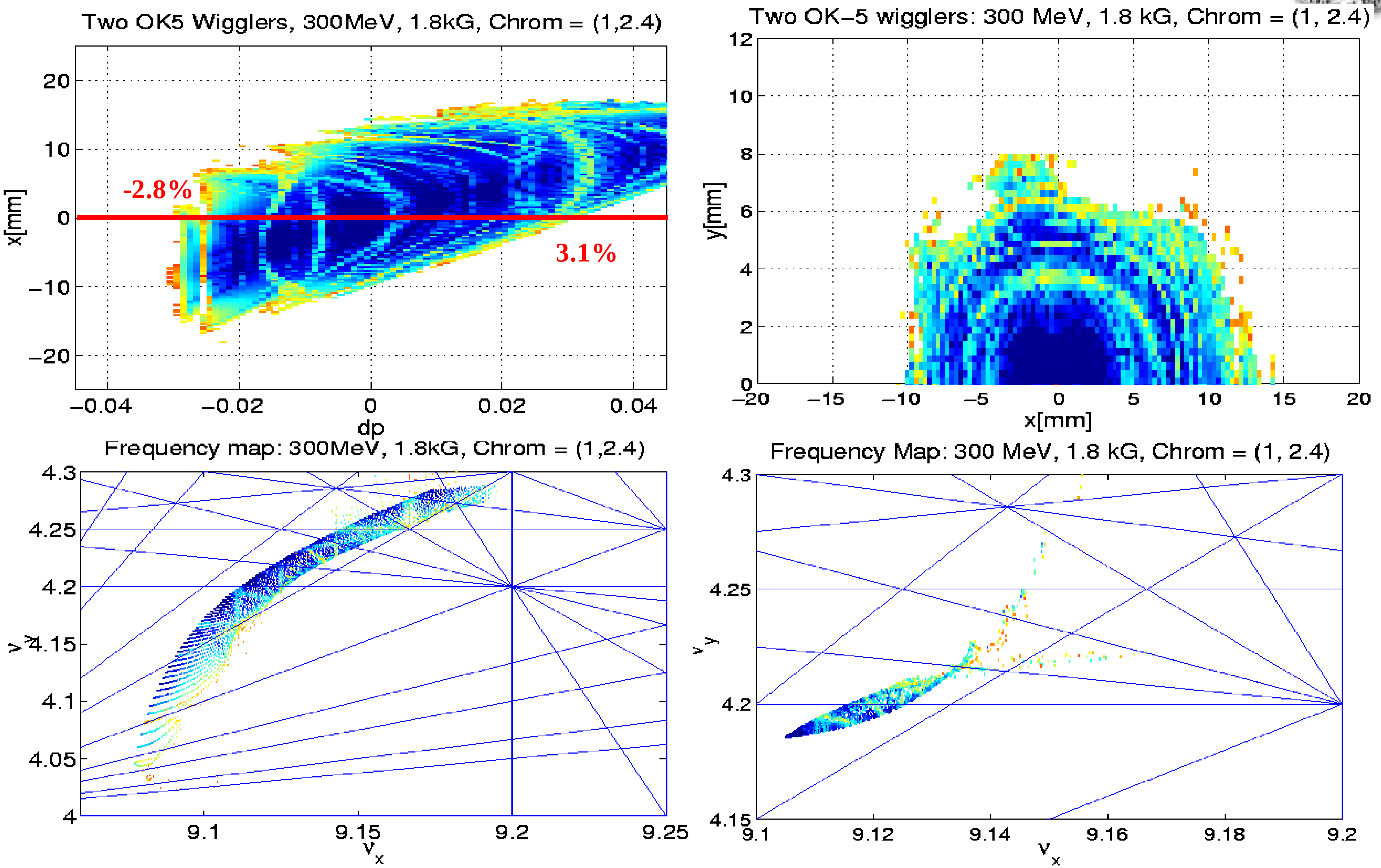




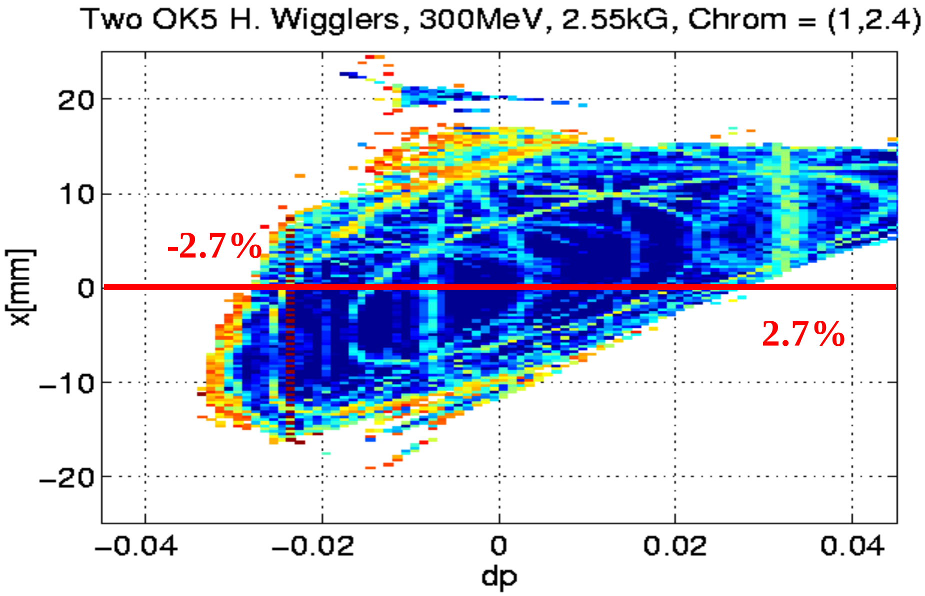


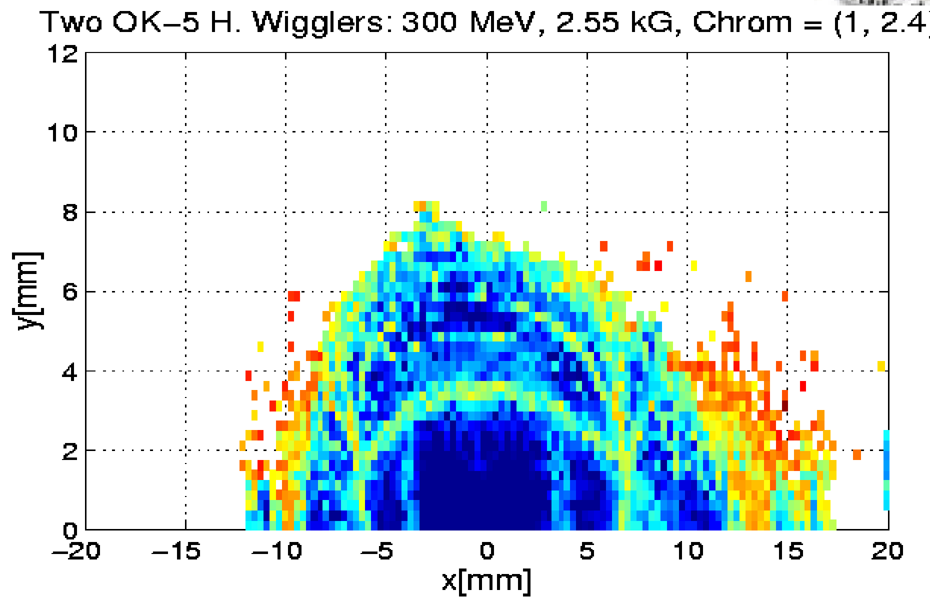


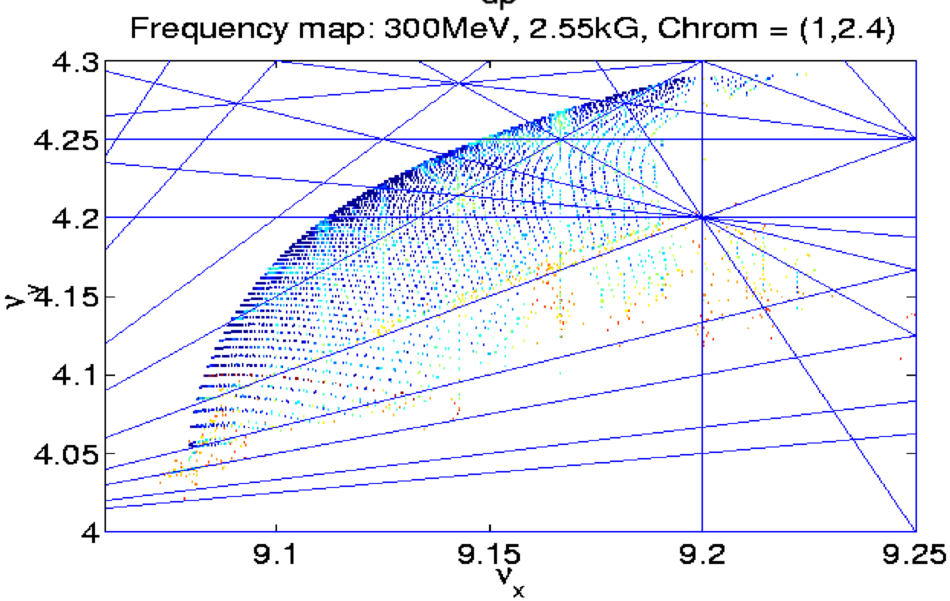
Dynamics with OK5 Wigglers: 300 MeV, 1.8 kG

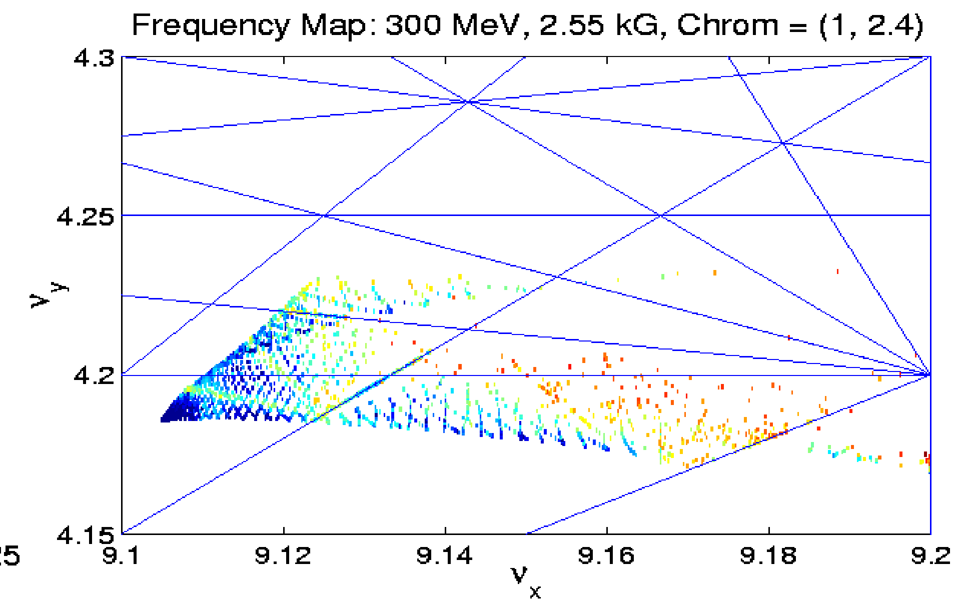


Dynamics with OK5 Wigglers: 300 MeV, 2.55 kG

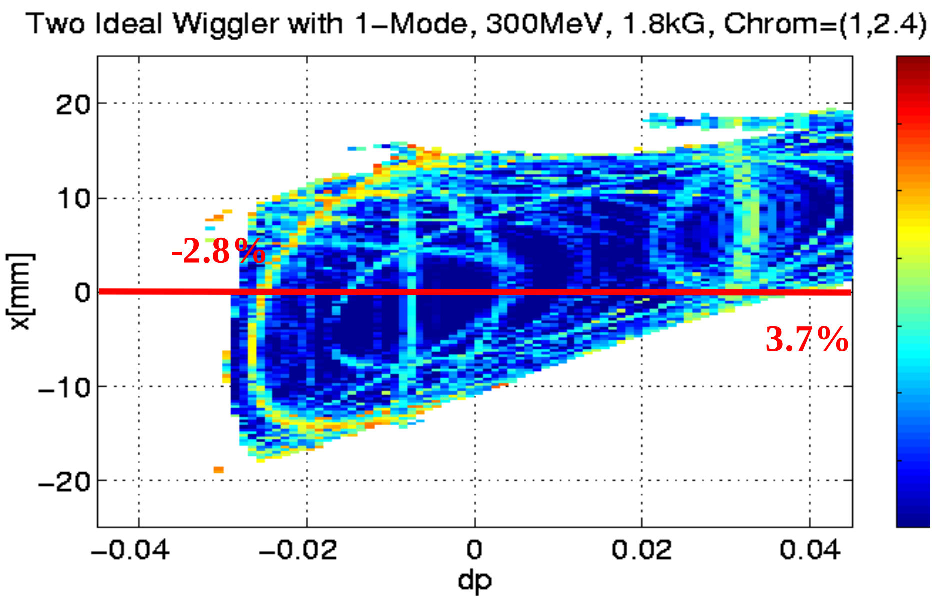


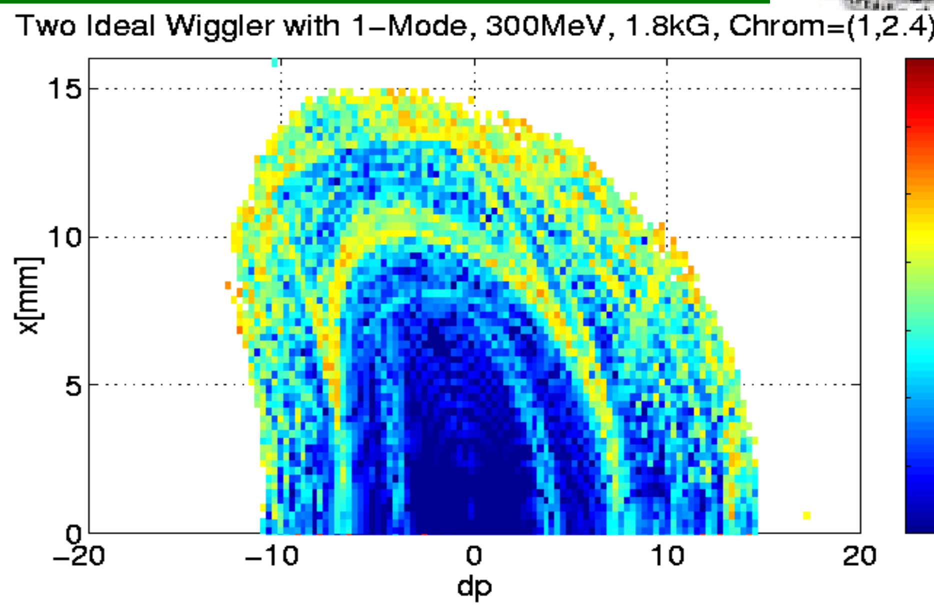


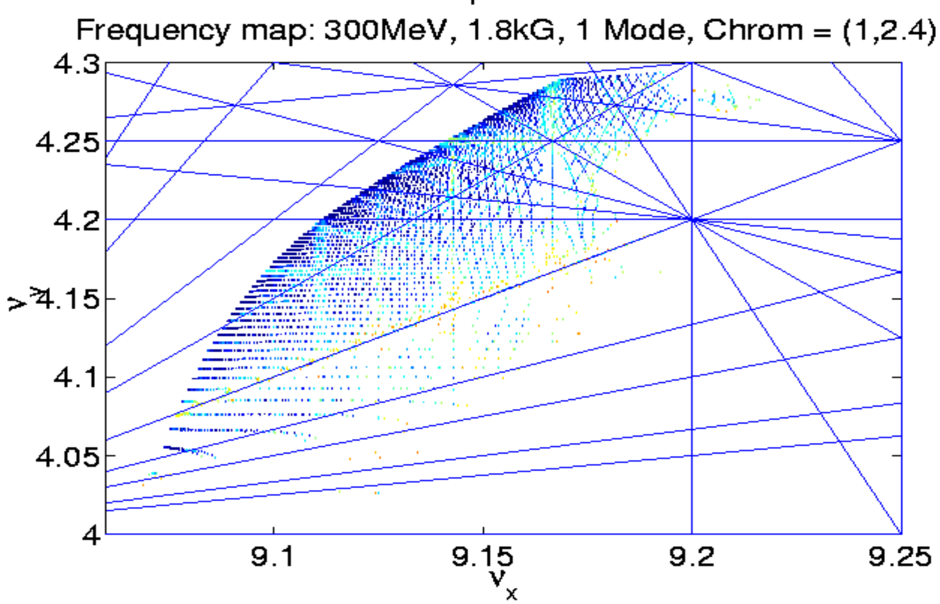


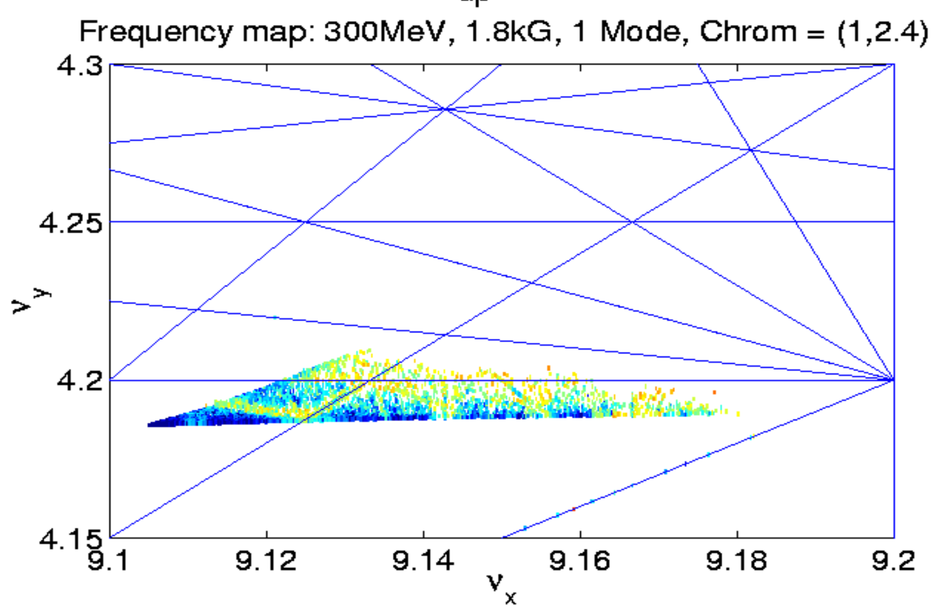


Dynamics with One-Mode Wigglers: 300 MeV, 1.8 kG









OK5 Wiggler Impacts on Dynamic Aperture (2 Wigglers)

Duke Storage Ring Chrom= (1,2.4)	*Eng Aperture (%)*	*Hori. Aperture (mm)*	*Vert. Aperture (mm)*
Bare Lattice	**(-2.7, 3.8)**	**(-15, 24.9)**	**23.5**
1 GeV, 5KG, Circular	**(-2.7, 3.8)**	**(-12.6, 17.7)**	**8.86**
0.3 GeV, 1.8kG, Circular	**(-2.8, 3.1)**	**(-9.9, 13.5)**	**6.5**
0.3 GeV, 2.55kG, Linear	**(-2.7, 2.7)**	**(-11.7, 17.4)**	**7.25**
0.3 GeV, 1.8kG, Cir. (1mode)	**(-2.8, 3.7)**	**(-10.5, 14.7)**	**13.75**

- **Magnetic Wall Effects**

$$B_y = C_m \cos(k_x x)\cosh(k_y y)\cos(k_z z)$$

$$B_y \sim \left(\frac{K_w}{\gamma}\right)^2 k_x^4 x^4$$

$$\delta\nu \sim \left(\frac{K_w}{\gamma}\right)^2 k_x^4 \left(\int \beta_x^2 dz\right) J_x$$

- $(\mathbf{K_w}/\gamma)^2$ $\mathbf{L_w}$
- β_x^2
- $\mathbf{k_x}^4$

Summary: Applications and Future Work

Applications

- **Undulators and Wigglers: planar, circular, arbitrarily polarized:**
  Light Source Rings and Damping Rings;

- **3D Magnetic Fields: large aperture magnets or superconducting magnets**
  Hadron Colliders and Light Source Rings

- **Combined Solenoid and Focusing Magnets: final focusing system**
  Colliders

Future Work

- **3D Field modeling:**
  wiggler fields, fringe fields, and solenoid fields;

- **Treatment of Radiation Effects**

- **Symplectic Integrators for various complex 3D magnetic field elements**

Summary

- **OK5 FEL wigglers will have significant impact on beam dynamics in the Duke storage ring, especially at lower energies**
- **Field compensations can improve dynamic aperture**
- **Study of lattices with four OK5 wigglers is critical and under way**

References:

- **Explicit symplectic integrator for s-dependent static magnetic field,**

  Y. K. Wu, E. Forest, D. S. Robin, Phys. Rev. E 68, 046502 (2003)

- **Nonlinear Dynamics in the Duke Storage Ring with FEL Wigglers**

  Y. K. Wu, J. Li, S. F. Mikhailov, V. Litvinenko, PAC2003 (May 2003)