VLASOV-MAXWELL DESCRIPTION OF ELECTRON-PROTON (e-p) TWO-STREAM INSTABILITY IN HIGH-INTENSITY ACCELERATORS AND RINGS*

by

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Recent Publications on the Two-Steam Instability

- "Kinetic Description of Electron-Proton Instability in High-Intensity Proton Linacs and Storage Rings Based on the Vlasov-Maxwell Equations," R. C. Davidson, H. Qin, P. H. Stoltz, and T. -S. Wang, *Physical Review Special Topics on Accelerators and Beams* **2**, 054401 (1999).

- "Vlasov-Maxwell Description of Electron-Ion Two-Stream Instability in High-Intensity Linacs and Storage Rings," R. C. Davidson, H. Qin, and T. -S. Wang, *Physics Letters A* **252**, 213 (1999).

- "Kinetic Description of the Electron-Proton Instability in High-Intensity Linacs and Storage Rings," R. C. Davidson, H. Qin, W. W. Lee, and T. -S. Wang, *Proceedings of the 1999 Particle Accelerator Conference* **3**, 1623 (1999).

- "Multispecies Nonlinear Perturbative Particle Simulation of Intense Charged Particle Beam," H. Qin, R. C. Davidson, and W. W. Lee, *Proceedings of the 1999 Particle Accelerator Conference* **3**, 1626 (1999).

Theoretical Model and Assumptions

- Consider high-intensity ion beam with distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$, characteristic radius r_b, and directed axial momentum $\gamma_b m_b \beta_b c$, propagating in z-direction through background population of electrons with distribution function $f_e(\mathbf{x}, \mathbf{p}, t)$.

- Ions have high directed axial velocity $V_b = \beta_b c$, whereas electrons are nonrelativistic and stationary in the laboratory frame with $\int d^3p p_z f_e(\mathbf{x}, \mathbf{p}, t) \simeq 0$.

- Ion beam is treated as continuous in the z-direction, and applied transverse focusing force is modeled by

$$\mathbf{F}^b_{foc} = -\gamma_b m_b \omega^2_{\beta b} \mathbf{x}_\perp$$

in the smooth-beam approximation, where $\mathbf{x}_\perp = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ is transverse displacement from beam axis.

Theoretical Model and Assumptions

- For ion-rich beam, the space-charge force on an electron, $\mathbf{F}_e^s = e\nabla\phi$, provides transverse confinement of the electrons by the electrostatic potential $\phi(\mathbf{x}, t)$.

- Ion motion in the beam frame is assumed to be nonrelativistic, with

$$|p_x| \ , |p_y| \ |\delta p_z| \ll \gamma_b m_b \beta_b c$$

  where $\delta p_z = p_z - \gamma_b m_b \beta_b c$, and $\gamma_b m_b \beta_b c$ is the directed axial momentum.

- Allow arbitrary space-charge intensity consistent with transverse confinement of the ions by the focusing field.

Theoretical Model and Assumptions

- Analysis is carried out in the electrostatic approximation where the self-generated electric field is

$$\mathbf{E}^s(\mathbf{x}, t) = -\nabla\phi(\mathbf{x}, t)$$

- The electrostatic potential $\phi(x, y, z, t)$ is determined self-consistently from Poisson's equation

$$\nabla^2\phi = -4\pi e(Z_b n_b - n_e)$$

where $n_b(\mathbf{x}, t) = \int d^3p f_b(\mathbf{x}, \mathbf{p}, t)$ and $n_e = \int d^3p f_e(\mathbf{x}, \mathbf{p}, t)$ are the ion and electron number densities.

- Assume that the ion axial velocity profile $V_{zb}(\mathbf{x}, t) \simeq \beta_b c$ is approximately uniform over the beam cross section. The self-generated magnetic field

$$\mathbf{B}^s(\mathbf{x}, t) = \nabla A_z(\mathbf{x}, t) \times \hat{\mathbf{e}}_z$$

is determined from

$$\nabla^2 A_z = -4\pi Z_b e \beta_b n_b$$

where the electrons are assumed to carry zero axial current in the laboratory frame. (This assumption can be relaxed.)

Nonlinear Vlasov-Maxwell Equations

- In the context of these assumptions, the electron distribution $f_e(\mathbf{x}, \mathbf{p}, t)$ evolves nonlinearly according to

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + e\nabla\phi \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f_e(\mathbf{x}, \mathbf{p}, t) = 0$$

where $-e$ is the electron charge, and $\mathbf{v} = \mathbf{p}/m_e$.

- For the ions, the nonlinear Vlasov equation for $f_b(\mathbf{x}, \mathbf{p}, t)$ becomes

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - \left(\gamma_b m_b \omega_{\beta b}^2 \mathbf{x}_\perp + Z_b e \nabla_\perp \psi \right) \cdot \frac{\partial}{\partial \mathbf{p}_\perp} \right.$$

$$\left. - Z_b e \frac{\partial \phi}{\partial z} \frac{\partial}{\partial p_z} \right\} f_b(\mathbf{x}, \mathbf{p}, t) = 0$$

- Here, $\mathbf{v} = \mathbf{p}/\gamma_b m_b$ is the ion velocity, $+Z_b e$ is the ion charge, and $\psi(\mathbf{x}, t)$ is the combined potential defined by

$$\psi(\mathbf{x}, t) \equiv \phi(\mathbf{x}, t) - \beta_b A_z(\mathbf{x}, t)$$

Nonlinear Vlasov-Maxwell Equations

- The electrostatic potential $\phi(\mathbf{x},t)$ and combined potential $\psi(\mathbf{x},t) = \phi(\mathbf{x},t) - \beta_b A_z(\mathbf{x},t)$ are determined self-consistently from

$$\nabla^2\phi = -4\pi e\left(Z_b\int d^3p f_b - \int d^3p f_e\right)$$

$$\nabla^2\psi = -4\pi e\left(\frac{Z_b}{\gamma_b^2}\int d^3p f_b - \int d^3p f_e\right)$$

- In Maxwell's equations for $\phi(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$

$$n_b(\mathbf{x},t) = \int d^3p f_b(\mathbf{x},\mathbf{p},t)$$

$$n_e(\mathbf{x},t) = \int d^3p f_e(\mathbf{x},\mathbf{p},t)$$

are the ion and electron particle densities, respectively.

Theoretical Model and Assumptions

- Under equilibrium conditions ($\partial/\partial t = 0$), treat the ion and electron properties as spatially uniform in the z-direction ($\partial/\partial z = 0$).

- In the stability analysis, assume small-amplitude perturbations with $z-$ and $t-$variations of the form

$$\exp(ik_z z - i\omega t)$$

  where $Im\omega > 0$ corresponds to instability (temporal growth), and $k_z = 2\pi n/L$ is the axial wavenumber, where n is an integer, and L is the axial periodicity length of the perturbation. ($L = 2\pi R$ for a storage ring, where R is the ring radius.)

- Stability analysis assumes perturbations with sufficiently long axial wavelength that

$$k_z^2 r_b^2 \ll 1.$$

Theoretical Model and Assumptions

The assumption of long axial wavelength with $k_z^2 r_b^2 \ll 1$ leads to several simplifications in the analysis of the Vlasov-Maxwell equations.

- The three-dimensional Laplacian ∇^2 is approximated by

$$\nabla^2 \simeq \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- The perturbed axial forces on the electrons and ions, e.g.,

$$\delta \mathbf{F}_e = e \frac{\partial}{\partial z} \delta\phi \hat{\mathbf{e}}_z \quad \text{and} \quad \delta \mathbf{F}_b = -Z_b e \frac{\partial}{\partial z} \delta\phi \hat{\mathbf{e}}_z$$

  are treated as small in comparison with the transverse forces.

- Two-stream instability will be strongest for high-frequency perturbations and small axial momentum spreads satisfying

$$\left| \frac{\omega}{k_z} - \beta_b c \right| \gg v_{Tbz}$$

$$\left| \frac{\omega}{k_z} \right| \gg v_{Tez}$$

  where $v_{Tbz} = (2T_{bz}/\gamma_b m_b)^{1/2}$ and $v_{Tez} = (2T_{ez}/m_e)^{1/2}$ are the characteristic axial thermal speeds.

Nonlinear Vlasov-Maxwell Equations

- Assume that a perfectly conducting cylindrical wall is located at radius $r = r_w$, where $r = (x^2 + y^2)^{1/2}$. Impose the requirement that

$$[E_\theta^s]_{r=r_w} = [E_z^s]_{r=r_w} = [B_r^s]_{r=r_w} = 0$$

- In terms of the potentials $\phi(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$, this gives

$$\phi(r = r_w, \theta, z, t) = 0$$

$$\psi(r = r_w, \theta, z, t) = 0$$

where the constant values of the potentials at $r = r_w$ have been taken equal to zero without loss of generality.

Equilibrium Vlasov-Maxwell Equations

- Under quasisteady conditions, examine solutions to nonlinear Vlasov-Maxwell equations with

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0$$

- Vlasov-Maxwell equations support broad range of equilibrium solutions for the beam ions and background electrons of the general form

$$f_b^0(r, \mathbf{p}) = F_b(H_{\perp b})G_b(p_z)$$

$$f_e^0(r, \mathbf{p}) = F_e(H_{\perp e})G_e(p_z)$$

- Here, $H_{\perp b}$ and $H_{\perp e}$ are the single-particle Hamiltonians defined by

$$H_{\perp b} = \frac{1}{2\gamma_b m_b}\mathbf{p}_\perp^2 + \frac{1}{2}\gamma_b m_b \omega_{\beta b}^2 r^2 + Z_b e[\psi^0(r) - \hat{\psi}^0]$$

$$H_{\perp e} = \frac{1}{2m_e}\mathbf{p}_\perp^2 - e[\phi^0(r) - \hat{\phi}^0]$$

where $r = (x^2 + y^2)^{1/2}$, and the constants $\hat{\phi}^0 \equiv \phi^0(r = 0)$ and $\hat{\psi}^0 \equiv \psi^0(r = 0)$ are the on-axis values of the potentials.

Equilibrium Vlasov-Maxwell Equations

- The axial momentum distributions $G_j(p_z)$ $(j = b, e)$ are normalized according to

$$\int_{-\infty}^{\infty} dp_z G_j(p_z) = 1$$

where $G_b(p_z)$ is centered at $p_z \simeq \gamma_b m_b V_b$ and $G_e(p_z)$ is centered at $p_z \simeq 0$.

- Many choices of $G_j(p_z)$ $(j = b, e)$ are possible. One example (for the beam ions) is the resonance distribution

$$G_b(p_z) = \frac{\Delta_b}{\pi[(p_z - \gamma_b m_b V_b)^2 + \Delta_b^2]},$$

where $\Delta_b = const.$ is a measure of the axial momentum spread,

Equilibrium Vlasov-Maxwell Equations

- For specified transverse distribution functions $F_b(H_{\perp b})$ and $F_e(H_{\perp e})$, the equilibrium potentials $\phi^0(r)$ and $\psi^0(r)$ are determined self-consistently from

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\phi^0(r) = -4\pi e[Z_b n_b^0(r) - n_e^0(r)]$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\psi^0(r) = -4\pi e\left[\frac{Z_b}{\gamma_b^2}n_b^0(r) - n_e^0(r)\right]$$

where $n_b^0(r)$ and $n_e^0(r)$ are the ion and electron density profiles

$$n_b^0(r) = \int d^3p F_b(H_{\perp b})G_b(p_z)$$

$$n_e^0(r) = \int d^3p F_e(H_{\perp e})G_e(p_z)$$

- Maxwell's equations for $\phi^0(r)$ and $\psi^0(r)$ are generally *nonlinear*.

Equilibrium with Step-Function Density Profiles

- A simple class of equilibrium distribution functions $F_b(H_{\perp b})$ and $F_e(H_{\perp e})$, which correspond to overlapping *step-function* density profiles for the beam ions and background electrons, is given by

$$F_b(H_{\perp b}) = \frac{\widehat{n}_b}{2\pi\gamma_b m_b}\delta(H_{\perp b} - \widehat{T}_{\perp b})$$

$$F_e(H_{\perp e}) = \frac{\widehat{n}_e}{2\pi m_e}\delta(H_{\perp e} - \widehat{T}_{\perp e})$$

where $\widehat{n}_b$, $\widehat{n}_e$, $\widehat{T}_{\perp b}$, and $\widehat{T}_{\perp e}$ are positive constants.

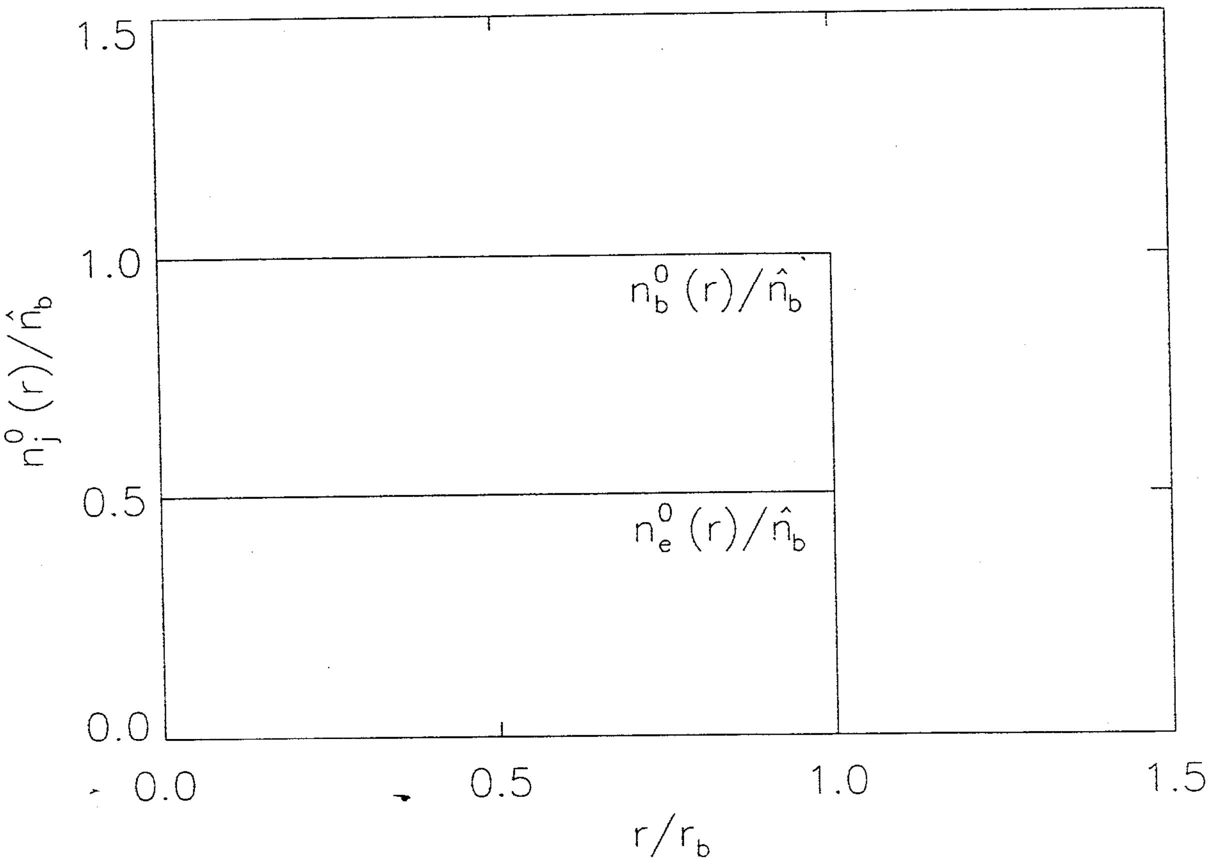
- Some straightforward algebraic manipulation shows that the corresponding density profiles are

$$n_b^0(r) = \begin{cases} \widehat{n}_b = const., & 0 \le r < r_b \\ 0\ , & r_b < r \le r_w \end{cases}$$

and

$$n_e^0(r) = \begin{cases} \widehat{n}_e \equiv f Z_b \widehat{n}_b = const., & 0 \le r < r_b \\ 0\ , & r_b < r \le r_w \end{cases}$$

where $f \equiv \widehat{n}_e / Z_b \widehat{n}_b$ is the fractional charge neutralization.



1.5
1.0
0.5
0.0
$n_j^0(r)/\hat{n}_b$
$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
0.0
0.5
1.0
1.5
r/r_b

Equilibrium with Step-Function Density Profiles

- Introduce the ion plasma frequency-squared defined by

$$\widehat{\omega}_{pb}^2 \equiv \frac{4\pi \widehat{n}_b Z_b^2 e^2}{\gamma_b m_b} = \frac{4 N_b Z_b^2 e^2}{\gamma_b m_b r_b^2}$$

where $N_b = \pi \widehat{n}_b r_b^2$ is the number of beam ions per unit axial length.

- Equilibrium analysis shows that the beam radius r_b is related to $\widehat{T}_{\perp b}$, $\widehat{T}_{\perp e}$, $\widehat{\omega}_{pb}^2$, etc., by the equilibrium constraint conditions

$$\left[\omega_{\beta b}^2 - \frac{1}{2}\left(\frac{1}{\gamma_b^2} - f\right)\widehat{\omega}_{pb}^2\right] r_b^2 = \frac{2\widehat{T}_{\perp b}}{\gamma_b m_b}$$

$$\frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}\left(1 - f\right)\widehat{\omega}_{pb}^2 \, r_b^2 = \frac{2\widehat{T}_{\perp e}}{m_e}$$

- The coefficients of r_b^2 in the above constraint conditions will be recognized as the *depressed* betatron frequencies

$$\widehat{\nu}_b^2 \quad \text{and} \quad \widehat{\nu}_e^2$$

for transverse particle motions, including self-field effects.

Equilibrium with Step-Function Density Profiles

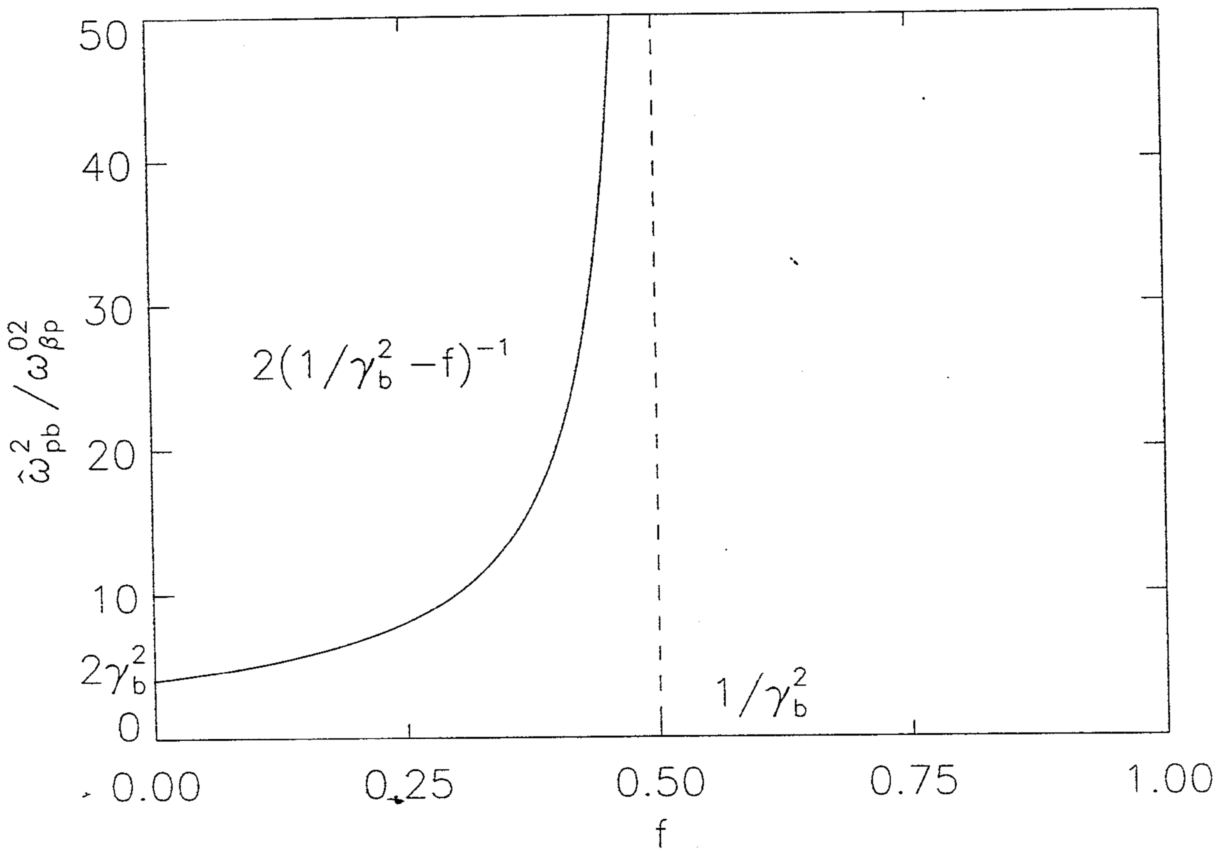
- Examine equilibrium constraint conditions for $\widehat{T}_{\perp b} \geq 0$ and $\widehat{T}_{\perp e} \geq 0$.

- Can show that both the ions and electrons are radially confined provided

$$f < 1$$

$$\frac{1}{2}\frac{\widehat{\omega}_{pb}^2}{\omega_{\beta b}^2}\left(\frac{1}{\gamma_b^2} - f\right) < 1$$

which place restrictions on the allowed values of fractional charge neutralization f, and normalized beam intensity $\widehat{\omega}_{pb}^2/\omega_{\beta b}^2$.



50
40
30
20
10
0
$2\gamma_b^2$
$\hat{\omega}_{pb}^2/\omega_{\beta p}^{02}$
$2(1/\gamma_b^2 - f)^{-1}$
$1/\gamma_b^2$
0.00
0.25
0.50
0.75
1.00
f

Thermal Equilibrium with Diffuse Density Profiles

- Many choices of equilibrium distributions $F_b(H_{\perp b})$ and $F_e(H_{\perp e})$ are possible. As another example, consider

$$F_b(H_{\perp b}) = \frac{\widehat{n}_b}{(2\pi \gamma_b m_b T_{\perp b})} \exp\left(-\frac{H_{\perp b}}{T_{\perp b}}\right)$$

$$F_b(H_{\perp e}) = \frac{\widehat{n}_e}{(2\pi m_e T_{\perp e})} \exp\left(-\frac{H_{\perp e}}{T_{\perp e}}\right)$$

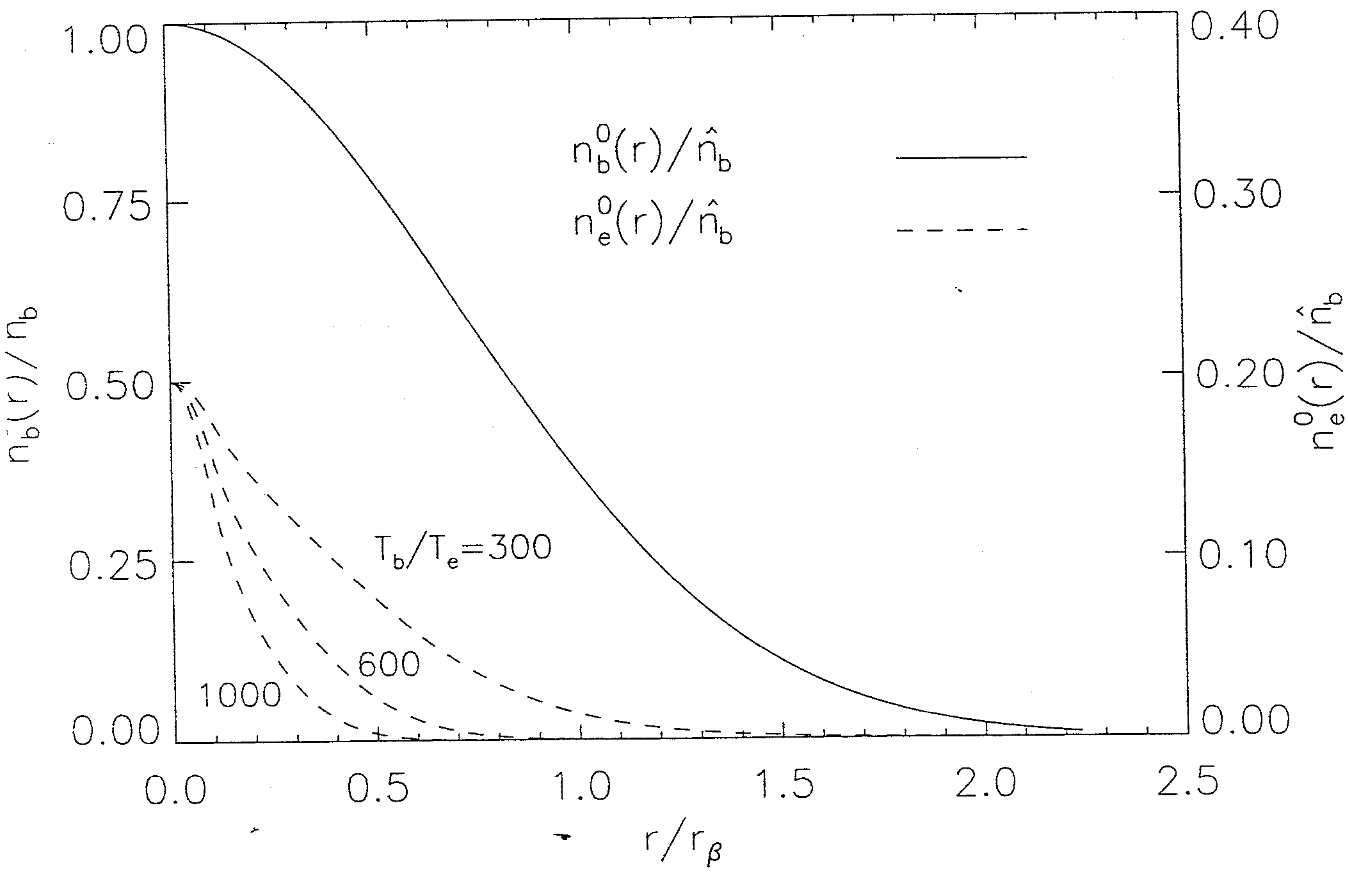
where $\widehat{n}_b$, $\widehat{n}_e$, $T_{\perp b}$, and $T_{\perp e}$ are positive constants.

- The corresponding equilibrium density profiles are

$$n_b^0(r) = \widehat{n}_b \exp\left\{-\frac{1}{T_{\perp b}}\left(\frac{1}{2}\gamma_b m_b \omega_{\beta b}^2 r^2 + Z_b e[\psi^0(r) - \widehat{\psi}^0]\right)\right\}$$

$$n_e^0(r) = \widehat{n}_e \exp\left\{\frac{e}{T_{\perp e}}[\phi^0(r) - \widehat{\phi}^0]\right\}$$

- The potentials $\psi^0(r)$ and $\phi^0(r)$ must be determined numerically from the corresponding Maxwell equations, which are highly nonlinear.



1.00
0.75
0.50
0.25
0.00
$n_b(r)/n_b$
0.40
0.30
0.20
0.10
0.00
$n_e^0(r)/\hat{n}_b$
$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
T_b/T_e=300
600
1000
0.0
0.5
1.0
1.5
2.0
2.5
r/r_β

Linearized Vlasov-Maxwell Equations

- Express all quantities in the nonlinear Vlasov-Maxwell equations as an equilibrium value plus a perturbation, e.g., $f_b(\mathbf{x},\mathbf{p},t) = f_b^0(r,\mathbf{p}) + \delta f_b(\mathbf{x},\mathbf{p},t)$, $\psi(\mathbf{x},t) = \psi^0(r) + \delta\psi(\mathbf{x},t)$, etc.

- For small-amplitude perturbations, the linearized Vlasov equation for the ions becomes

$$
\begin{aligned}
&\left\{\frac{\partial}{\partial t} + \frac{p_z}{\gamma_b m_b}\frac{\partial}{\partial z} + \frac{\mathbf{p}_\perp}{\gamma_b m_b}\cdot\frac{\partial}{\partial \mathbf{x}_\perp}\right.\\
&\left. - \left[\gamma_b m_b \omega_{\beta b}^2 + \frac{Z_b e}{r}\frac{\partial}{\partial r}\psi^0(r)\right]\mathbf{x}_\perp\cdot\frac{\partial}{\partial \mathbf{p}_\perp}\right\}\delta f_b(\mathbf{x},\mathbf{p},t)\\
&= \frac{Z_b e}{\gamma_b m_b}\mathbf{p}_\perp\cdot\nabla_\perp\delta\psi(\mathbf{x},t)\frac{\partial}{\partial H_{\perp b}}F_b(H_{\perp b})G_b(p_z)
\end{aligned}
$$

Linearized Vlasov-Maxwell Equations

- Similarly, the linearized Vlasov equation for the electrons is given by

$$\left\{\frac{\partial}{\partial t}+\frac{p_z}{m_e}\frac{\partial}{\partial z}+\frac{\mathbf{p}_\perp}{m_e}\cdot\frac{\partial}{\partial \mathbf{x}_\perp}+\frac{e}{r}\frac{\partial}{\partial r}\phi^0(r)\mathbf{x}_\perp\cdot\frac{\partial}{\partial \mathbf{p}_\perp}\right\}$$

$$\times \quad \delta f_e(\mathbf{x},\mathbf{p},t)$$

$$= \quad -\frac{e}{m_e}\mathbf{p}_\perp\cdot\nabla_\perp\delta\phi(\mathbf{x},t)\frac{\partial}{\partial H_{\perp e}}F_e(H_{\perp e})G_e(p_z)$$

- Linearized Vlasov-Maxwell equations are valid for small-amplitude perturbations about general choice of equilibrium distribution functions $F_b(H_{\perp b})$ and $F_e(H_{\perp e})$.

Linearized Vlasov-Maxwell Equations

- The perturbed potentials $\delta\psi(\mathbf{x},t)$ and $\delta\phi(\mathbf{x},t)$ are determined self-consistently in terms of the perturbed distribution functions from the Maxwell equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta\psi = -4\pi e\left(\frac{Z_b}{\gamma_b^2}\int d^3p\delta f_b - \int d^3p\delta f_e\right)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta\phi = -4\pi e\left(Z_b\int d^3p\delta f_b - \int d^3p\delta f_e\right)$$

- In the linearized Vlasov equations for $\delta f_b(\mathbf{x},\mathbf{p},t)$ and $\delta f_e(\mathbf{x},\mathbf{p},t)$, it is important to recognize that the differential operator

$$\{\cdots\} = \frac{d}{dt'}$$

corresponds to the *total time derivative* following the particle motion in the total equilibrium (applied plus self-generated) field configuration.

Linearized Vlasov-Maxwell Equations

- For amplifying perturbations, we integrate the linearized Vlasov equations from $t' = -\infty$, where the perturbations are negligible small, up to the present time $t' = t$, when the particle orbits $\mathbf{x}'(t')$ and $\mathbf{p}'(t')$ pass through the phase-space point $(\mathbf{x}, \mathbf{p})$, i.e.,

$$\begin{aligned} \mathbf{x}'(t' = t) &= \mathbf{x} \\ \mathbf{p}'(t' = t) &= \mathbf{p} \end{aligned}$$

- This gives for the perturbed distribution functions

$$\delta f_b(\mathbf{x}, \mathbf{p}, t) = Z_b e \frac{\partial}{\partial H_{\perp b}} F_b(H_{\perp b}) G_b(p_z) \int_{-\infty}^{t} dt' \frac{\mathbf{p}'_\perp}{\gamma_b m_b} \cdot \nabla'_\perp \delta\psi(\mathbf{x}', t')$$

$$\delta f_e(\mathbf{x}, \mathbf{p}, t) = -e \frac{\partial}{\partial H_{\perp e}} F_e(H_{\perp e}) G_e(p_z) \int_{-\infty}^{t} dt' \frac{\mathbf{p}'_\perp}{m_e} \cdot \nabla'_\perp \delta\phi(\mathbf{x}', t')$$

where use has been made of $dH'_{\perp b}/dt' = 0 = dH'_{\perp e}/dt'$.

Linearized Vlasov-Maxwell Equations

- The 'primed' orbits for the beam ions solve $z'(t') = z + (p_z/\gamma_b m_b)(t' - t)$ and

$$\frac{d}{dt'}\mathbf{x}'_\perp(t') = \frac{1}{\gamma_b m_b}\mathbf{p}'_\perp(t')$$

$$\frac{d}{dt'}\mathbf{p}'_\perp(t') = -\gamma_b m_b \omega_{\beta b}^2 \mathbf{x}'_\perp(t') - \frac{Z_b e}{r'}\frac{\partial \psi^0(r')}{\partial r'}\mathbf{x}'_\perp(t')$$

where $r'^2(t') = x'^2(t') + y'^2(t')$. Similarly, the 'primed' orbits for the background electrons solve $z'(t') = z + (p_z/m_e)(t' - t)$, and

$$\frac{d}{dt'}\mathbf{x}'_\perp(t') = \frac{1}{m_e}\mathbf{p}'_\perp(t')$$

$$\frac{d}{dt'}\mathbf{p}'_\perp(t') = \frac{e}{r'}\frac{\partial \phi^0(r')}{\partial r'}\mathbf{x}'_\perp(t')$$

where $\mathbf{x}'_\perp(t' = t) = \mathbf{x}_\perp$ and $\mathbf{p}'_\perp(t' = t) = \mathbf{p}_\perp$.

Linearized Vlasov-Maxwell Equations

- The orbit integral representations for $\delta f_b(\mathbf{x}, \mathbf{p}, t)$ and $\delta f_e(\mathbf{x}, \mathbf{p}, t)$ must of course be substituted into the Maxwell equations for $\delta\psi(\mathbf{x}, t)$ and $\delta\phi(\mathbf{x}, t)$ to determine the self-consistent evolution of the perturbations.

- It is convenient to adopt a normal-mode approach in which perturbed quantities are expressed as

$$\delta f_b(\mathbf{x}, \mathbf{p}, t) = \sum_{\ell=-\infty}^{\infty} \sum_{k_z=-\infty}^{\infty} \delta \hat{f}_b^{\ell}(r, \mathbf{p}, k_z, \omega) \times \exp[i(\ell\theta + k_z z - \omega t)]$$

$$\delta\psi(\mathbf{x}, t) = \sum_{\ell=-\infty}^{\infty} \sum_{k_z=-\infty}^{\infty} \delta\hat{\psi}^{\ell}(r, k_z, \omega) \times \exp[i(\ell\theta + k_z z - \omega t)]$$

Linearized Vlasov-Maxwell Equations

- Here, $(x, y) = (r\cos\theta, r\sin\theta)$ is the transverse position, the integer ℓ is the azimuthal mode number, $k_z = 2\pi n/L$ is the axial wavenumber, where n is an integer and L is the axial periodicity length, and ω is the complex oscillation frequency.

- When carrying out the t'-integration, $Im\omega > 0$ is assumed, corresponding to instability (temporal growth).

- In linac geometry, L is the fundamental periodicity length for Fourier analysis of the perturbations in the z-direction. In storage ring geometry, we make the identification $L = 2\pi R$, where R is the major radius of the storage ring ($R \gg r_b$).

Linearized Vlasov-Maxwell Equations

- Some straightforward algebra gives

$$
\begin{aligned}
\delta\widehat{f}_b^{\ell}(r,\mathbf{p}) &= Z_b e \frac{\partial}{\partial H_{\perp b}} F_b(H_{\perp b}) G_b(p_z) \\
&\times \left\{ \delta\widehat{\psi}^{\ell}(r) + i(\omega - k_z v_z) \int_{-\infty}^{0} d\tau \delta\widehat{\psi}^{\ell}(r') \right. \\
&\times \left. \exp[i\ell(\theta' - \theta) - i(\omega - k_z v_z)\tau] \right\}
\end{aligned}
$$

for the beam ions, where $v_z = p_z/\gamma_b m_b$, and

$$
\begin{aligned}
\delta\widehat{f}_e^{\ell}(r,\mathbf{p}) &= -e \frac{\partial}{\partial H_{\perp e}} F_e(H_{\perp e}) G_e(p_z) \\
&\times \left\{ \delta\widehat{\phi}^{\ell}(r) + i(\omega - k_z v_z) \int_{-\infty}^{0} d\tau \delta\widehat{\phi}^{\ell}(r') \right. \\
&\times \left. \exp[i\ell(\theta' - \theta) - i(\omega - k_z v_z)\tau] \right\}
\end{aligned}
$$

for the background electrons, where $v_z = p_z/m_e$.

Linearized Vlasov-Maxwell Equations

- Here, $Im\omega > 0$ is assumed, and τ denotes the displaced time variable

$$\tau = t' - t$$

- The radial and azimuthal orbits, $r'(t')$ and $\theta'(t')$, satisfy

$$r'(t' = t) = r$$

$$\theta'(t' = t) = \theta$$

and are related to the Cartesian orbits, $x'(t')$ and $y'(t')$, by $x' = r' \cos\theta'$ and $y' = r' \sin\theta'$.

Linearized Vlasov-Maxwell Equations

- Finally, for self-consistency of the perturbed fields, Maxwell's equations for $\delta\widehat{\psi}^\ell(r)$ and $\delta\widehat{\phi}^\ell(r)$ can be expressed as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{\ell^2}{r^2}\right)\delta\widehat{\psi}^\ell(r)$$

$$= \quad -4\pi e\left(\frac{Z_b}{\gamma_b^2}\int d^3p\delta\widehat{f}_b^\ell(r,\mathbf{p}) - \int d^3p\delta\widehat{f}_e^\ell(r,\mathbf{p})\right)$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{\ell^2}{r^2}\right)\delta\widehat{\phi}^\ell(r)$$

$$= \quad -4\pi e\left(Z_b\int d^3p\delta\widehat{f}_b^\ell(r,\mathbf{p}) - \int d^3p\delta\widehat{f}_e^\ell(r,\mathbf{p})\right)$$

- The four coupled equations for $\delta\widehat{f}_b^\ell(r,\mathbf{p})$, $\delta\widehat{f}_e^\ell(r,\mathbf{p})$, $\delta\widehat{\psi}^\ell(r)$ and $\delta\widehat{\phi}^\ell(r)$ represent the final system of eigenvalue equations derived for small-amplitude perturbations about *general* equilibrium distributions $f_j^0(r,\mathbf{p}) = F_j(H_{\perp j})G_j(p_z)$.

Linearized Vlasov-Maxwell Equations

- The coupled eigenvalue equations have a wide range of applicability, and can be used to determine the complex oscillation frequency ω and detailed stability properties for a wide range of system parameters and choices of transverse distribution functions $F_b(H_{\perp b})$ and $F_e(H_{\perp e})$.

- The principal challenge in analyzing the coupled eigenvalue equations is two-fold:

  (a) Depending on the equilibrium profiles, the transverse orbits (r', θ') or (x', y') are often difficult to calculate in closed analytical form.

  (b) Once the orbits in the equilibrium fields are determined, the integrals over t' are challenging because the r'-orbits occur explicitly in the arguments of the (yet unknown) eigenfunction amplitudes $\delta\widehat{\phi}^\ell(r')$ and $\delta\widehat{\psi}^\ell(r')$.

Linearized Vlasov-Maxwell Equations

- For future reference, the ion and electron orbit equations can be expressed in the convenient forms

$$\frac{d^2}{dt'^2}\mathbf{x}'_\perp(t') + \nu_b^2(r')\mathbf{x}'_\perp(t') = 0$$

and

$$\frac{d^2}{dt'^2}\mathbf{x}'_\perp(t') + \nu_e^2(r')\mathbf{x}'_\perp(t') = 0$$

Here, $\nu_b^2(r)$ and $\nu_e^2(r)$ are the (depressed) betatron frequencies-squared, including applied plus self-field effects:

$$\nu_b^2(r) = \omega_{\beta b}^2 + \frac{Z_b e}{\gamma_b m_b}\frac{1}{r}\frac{\partial}{\partial r}\psi^0(r)$$

$$\nu_e^2(r) = -\frac{e}{m_e}\frac{1}{r}\frac{\partial}{\partial r}\phi^0(r)$$

Particle Orbits for Step-Function Density Profiles

- For step-function density profiles, the (depressed) betatron frequencies are given exactly in the beam interior ($0 \leq r < r_b$) by

$$\nu_b^2(r) = \hat{\nu}_b^2 \equiv \omega_{\beta b}^2 - \frac{1}{2}\hat{\omega}_{pb}^2\left(\frac{1}{\gamma_b^2} - f\right)$$

$$\nu_e^2(r) = \hat{\nu}_e^2 \equiv \frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}\hat{\omega}_{pb}^2(1-f)$$

where $\hat{\nu}_b^2$ and $\hat{\nu}_e^2$ are constants (independent of r), and $\hat{\omega}_{pb}^2 = 4\pi\hat{n}_b Z_b^2 e^2/\gamma_b m_b$.

- The ion orbits that pass through (x, y, p_x, p_y) at time $t' = t$ are (for $0 \leq r' < r_b$)

$$x'(t') = x\cos(\hat{\nu}_b\tau) + \frac{1}{\gamma_b m_b \hat{\nu}_b} p_x \sin(\hat{\nu}_b\tau)$$

$$y'(t') = y\cos(\hat{\nu}_b\tau) + \frac{1}{\gamma_b m_b \hat{\nu}_b} p_y \sin(\hat{\nu}_b\tau)$$

Particle Orbits for Step-Function Density Profiles

- Similarly, the electron orbits are given by (for $0 \leq r' < r_b$)

$$x'(t') = x\cos(\widehat{\nu}_e\tau) + \frac{1}{m_e\widehat{\nu}_e}p_x\sin(\widehat{\nu}_e\tau)$$

$$y'(t') = y\cos(\widehat{\nu}_e\tau) + \frac{1}{m_e\widehat{\nu}_e}p_y\sin(\widehat{\nu}_e\tau)$$

- Representation of the orbits (r', θ') in cylindrical coordinates is also readily obtained, where $x' = r'\cos\theta'$ and $y' = r'\sin\theta'$. We introduce

$$p_x = p_\perp\cos\phi$$

$$p_y = p_\perp\sin\phi$$

where ϕ is the azimuthal phase of $\mathbf{p}_\perp$. It then follows that $r'^2(t') = x'^2(t') + y'^2(t')$ can be expressed (for the ions, say) as

$$r'^2(t') = \frac{1}{2}r^2[1 + \cos(2\widehat{\nu}_b\tau)] + \frac{p_\perp^2}{2\gamma_b^2 m_b^2\widehat{\nu}_b^2}[1 - \cos(2\widehat{\nu}_b\tau)]$$

$$+ \frac{rp_\perp}{\gamma_b m_b\widehat{\nu}_b}\cos(\phi - \theta)\sin(2\widehat{\nu}_b\tau)$$

Dispersion Relation for Step-Function Density Profiles

- Express $\int d^3p \cdots = \int_0^\infty dp_\perp p_\perp \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dp_z \cdots$ in calculations of $\int d^3p \delta \widehat{f}_b$ and $\int d^3p \delta \widehat{f}_e$. Because $\partial F_b(H_{\perp b})/\partial H_{\perp b}$ and $\partial F_e(H_{\perp e})/\partial H_{\perp e}$ are independent of azimuthal momentum phase ϕ, what is required is the phase-averaged orbit integrals

$$
\begin{aligned}
I_b^\ell(r, p_\perp, p_z) &= i(\omega - k_z v_z) \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-\infty}^{0} d\tau \delta \widehat{\psi}_\ell(r') \\
&\times \exp\{i\ell(\theta' - \theta) - i(\omega - k_z v_z)\tau\}
\end{aligned}
$$

$$
\begin{aligned}
I_e^\ell(r, p_\perp, p_z) &= i(\omega - k_z v_z) \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-\infty}^{0} d\tau \delta \widehat{\phi}_\ell(r') \\
&\times \exp\{i\ell(\theta' - \theta) - i(\omega - k_z v_z)\tau\}
\end{aligned}
$$

- For step-function density profiles, a class of solutions is permitted in which

$$
\begin{aligned}
\delta \widehat{\psi}^\ell(r) &= \widehat{\psi}_\ell r^\ell \\
\delta \widehat{\phi}^\ell(r) &= \widehat{\phi}_\ell r^\ell
\end{aligned}
$$

in the beam interior ($0 \le r < r_b$). Here, $\widehat{\psi}_\ell$ and $\widehat{\phi}_\ell$ are constant amplitudes.

Dispersion Relation for Step-Function Density Profiles

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- We make use of

$$(x' + iy')^\ell = r'^\ell \exp(i\ell\theta')$$

to express

$$I_b^\ell(r, p_\perp, p_z) = i(\omega - k_z v_z)\widehat{\psi}_\ell \exp(-i\ell\theta)$$

$$\times \int_{-\infty}^{0} d\tau \exp\{-i(\omega - k_z v_z)\tau\} \int_0^{2\pi} \frac{d\phi}{2\pi}[x'(t') + iy'(t')]^\ell$$

- Integrating over ϕ and τ gives (exactly)

$$I_b^\ell(r, p_\perp, p_z) = -\frac{1}{2^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}$$
$$\times \frac{(\omega - k_z v_z)}{[\omega - k_z v_z - (\ell - 2m)\widehat{\nu}_b]}\delta\widehat{\psi}^\ell(r)$$

for the beam ions, where $v_z = p_z/\gamma_b m_b$, and

$$I_e^\ell(r, p_\perp, p_z) = -\frac{1}{2^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}$$
$$\times \frac{(\omega - k_z v_z)}{[\omega - k_z v_z - (\ell - 2m)\widehat{\nu}_e]}\delta\widehat{\phi}^\ell(r)$$

for the background electrons, where $v_z = p_z/m_e$.

Dispersion Relation for Step-Function Density Profiles

- Some straightforward algebra gives for the perturbed charge densities

$$4\pi Z_b e \int d^3p\,\delta\widehat{f}_b^\ell(r,\mathbf{p})$$

$$= -\frac{\widehat{\omega}_{pb}^2}{\widehat{\nu}_b^2}\Gamma_b^\ell(\omega)\delta\widehat{\psi}^\ell(r)\frac{1}{r_b}\delta(r-r_b)$$

$$-4\pi e \int d^3p\,\delta\widehat{f}_e^\ell(r,\mathbf{p}) = -\frac{\widehat{\omega}_{pe}^2}{\widehat{\nu}_e^2}\Gamma_e^\ell(\omega)\delta\widehat{\phi}^\ell(r)\frac{1}{r_b}\delta(r-r_b)$$

- Here, the ion and electron response functions ($j = b, e$) are defined by

$$\Gamma_j^\ell(\omega) = -\frac{1}{2^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}\int_\infty^\infty dp_z\frac{(\ell-2m)\widehat{\nu}_j G_j(p_z)}{[(\omega-k_z v_z)-(\ell-2m)\widehat{\nu}_j]}$$

for general azimuthal mode number ℓ.

Dispersion Relation for Step-Function Density Profiles

- The coupled equations for the eigenfunction amplitudes $\delta\widehat{\psi}^\ell(r)$ and $\delta\widehat{\phi}^\ell(r)$ then become

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}-\frac{\ell^2}{r^2}\right)\delta\widehat{\psi}^\ell(r)$$

$$= \left[\frac{\widehat{\omega}_{pb}^2}{\gamma_b^2\widehat{\nu}_b^2}\Gamma_b^\ell(\omega)\delta\widehat{\psi}^\ell(r)+\frac{\widehat{\omega}_{pe}^2}{\widehat{\nu}_e^2}\Gamma_e^\ell(\omega)\delta\widehat{\phi}^\ell(r)\right]$$

$$\times \quad \frac{1}{r_b}\delta(r-r_b)$$

and

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}-\frac{\ell^2}{r^2}\right)\delta\widehat{\phi}^\ell(r)$$

$$= \left[\frac{\widehat{\omega}_{pb}^2}{\widehat{\nu}_b^2}\Gamma_b^\ell(\omega)\delta\widehat{\psi}^\ell(r)+\frac{\widehat{\omega}_{pe}^2}{\widehat{\nu}_e^2}\Gamma_e^\ell(\omega)\delta\widehat{\phi}^\ell(r)\right]$$

$$\times \quad \frac{1}{r_b}\delta(r-r_b)$$

Dispersion Relation for Step-Function Density Profiles

- The equations for $\delta\widehat{\psi}^{\ell}(r)$ and $\delta\widehat{\phi}^{\ell}(r)$ can be solved *exactly* in the beam interior $(0 \leq r < r_b)$, and in the vacuum region $(r_b < r \leq r_w)$.

- These equations can also be integrated across the beam surface at $r = r_b$, thereby relating the discontinuities in $(\partial/\partial r)\delta\widehat{\phi}^{\ell}(r)$ and $(\partial/\partial r)\delta\widehat{\psi}^{\ell}(r)$ self-consistently to the perturbed surface-charge and surface-current densities.

Dispersion Relation for Step-Function Density Profiles

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- We enforce continuity of $\delta\widehat{\phi}^\ell(r)$ and $\delta\widehat{\psi}^\ell(r)$ at the beam surface ($r = r_b$), and set $\delta\widehat{\phi}^\ell(r = r_w) = 0 = \delta\widehat{\psi}^\ell(r = r_w)$. Some straightforward algebra gives

$$\left[\frac{2}{1-(r_b/r_w)^{2\ell}} + \frac{\widehat{\omega}_{pb}^2}{\ell\gamma_b^2\widehat{\nu}_b^2}\Gamma_b^\ell(\omega)\right]$$

$$\times\left[\frac{2}{1-(r_b/r_w)^{2\ell}} + \frac{\widehat{\omega}_{pe}^2}{\ell\widehat{\nu}_e^2}\Gamma_e^\ell(\omega)\right] = \frac{\widehat{\omega}_{pe}^2}{\ell\widehat{\nu}_e^2}\cdot\frac{\widehat{\omega}_{pb}^2}{\ell\widehat{\nu}_b^2}\Gamma_e^\ell(\omega)\Gamma_b^\ell(\omega)$$

- Dispersion relation is valid for:
  - Step-function density profiles $n_b^0(r)$ and $n_e^0(r)$.
  - Arbitrary normalized beam intensity $(\widehat{\omega}_{pb}^2/\omega_{\beta b}^2)$.
  - Arbitrary fractional charge neutralization f.
  - Arbitrary azimuthal harmonic number ℓ.
  - Incorporates effects of axial momentum spread.

- System is fully stable in the absence of background electrons ($f = 0$).

Ion and Electron Response Functions

- For purpose of illustration, take $G_j(p_z)$ $(j = b, e)$ to be the resonance function

$$G_j(p_z) = \frac{\Delta_j}{\pi[(p_z - \gamma_j m_j V_j)^2 + \Delta_j^2]},$$

where $\Delta_j = const.$ is the characteristic axial momentum spread, $V_j = V_b$ for the beam ions, and $V_j = V_e = 0$ for the background electrons.

- The ion and electron response functions are then given by

$$\Gamma_b(\omega) = -\frac{1}{2^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}$$
$$\times \quad \frac{(\ell-2m)\widehat{\nu}_b}{[(\omega - k_z V_b + i|k_z|v_{Tbz}) - (\ell-2m)\widehat{\nu}_b]}$$

$$\Gamma_e(\omega) = -\frac{1}{2^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}$$
$$\times \quad \frac{(\ell-2m)\widehat{\nu}_e}{[(\omega + i|k_z|v_{Tez}) - (\ell-2m)\widehat{\nu}_e]},$$

where $v_{Tbz} = \Delta_b/\gamma_b m_b$ and $v_{Tez} = \Delta_e/m_e$ are the axial thermal speeds.

Dispersion Relation for Step-Function Density Profiles

- The contributions proportional $i|k_z|v_{Tjz}$ in the ion and electron response functions $\Gamma_j^l(\omega)$ correspond to Landau damping effects produced by longitudinal momentum spread.

- When two-stream instability occurs, the strongest instability (largest growth rate $Im\omega$) occurs for azimuthal mode number $\ell = 1$, corresponding to a simple (dipole) transverse displacement of the beam ions and background electrons.

Kinetic Dispersion Relation for Dipole Mode

- The exact kinetic dispersion relation for $\ell = 1$ can be expressed as

$$\left[\frac{2}{1-r_b^2/r_w^2} - \frac{\widehat{\omega}_{pb}^2/\gamma_b^2}{(\omega - k_z V_b + i|k_z|v_{Tzb})^2 - \widehat{\nu}_b^2}\right]$$

$$\times \left[\frac{2}{1-r_b^2/r_w^2} - \frac{\widehat{\omega}_{pe}^2}{(\omega + i|k_z|v_{Tze})^2 - \widehat{\nu}_e^2}\right]$$

$$= \frac{\widehat{\omega}_{pe}^2}{[(\omega + i|k_z|v_{Tze})^2 - \widehat{\nu}_e^2]} \frac{\widehat{\omega}_{pb}^2}{[(\omega - k_z V_b + i|k_z|v_{Tzb})^2 - \widehat{\nu}_b^2]}$$

- Here, we express

$$\widehat{\omega}_{pe}^2 = \frac{\gamma_b m_b}{Z_b m_e} f \widehat{\omega}_{pb}^2$$

where $\widehat{\omega}_{pb}^2 = 4\pi \widehat{n}_b Z_b^2 e^2/\gamma_b m_b$ is the ion plasma frequency-squared, and $f = \widehat{n}_e/Z_b\widehat{n}_b$ is the fractional charge neutralization by the background electrons.

Kinetic Dispersion Relation for Dipole Mode

- Kinetic dispersion relation for $\ell = 1$ can be expressed *exactly* in the compact form

$$[(\omega - k_z V_b + i|k_z|v_{Tzb})^2 - \omega_b^2][(\omega + i|k_z|v_{Tze})^2 - \omega_e^2] = \omega_f^4$$

- Here, the coupling frequency ω_f, and the ion and electron collective oscillation frequencies, ω_b and ω_e, are defined by

$$\omega_f^4 \equiv \frac{1}{4} f \left(1 - \frac{r_b^2}{r_w^2}\right)^2 \frac{\gamma_b m_b}{Z_b m_e} \widehat{\omega}_{pb}^4$$

$$\omega_b^2 \equiv \widehat{\nu}_b^2 + \frac{\widehat{\omega}_{pb}^2}{2\gamma_b^2} \left(1 - \frac{r_b^2}{r_w^2}\right) = \omega_{\beta b}^2 + \frac{1}{2}\widehat{\omega}_{pb}^2 \left(f - \frac{1}{\gamma_b^2}\frac{r_b^2}{r_w^2}\right)$$

and

$$\omega_e^2 \equiv \widehat{\nu}_e^2 + \frac{1}{2}\widehat{\omega}_{pe}^2 \left(1 - \frac{r_b^2}{r_w^2}\right) = \frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}\widehat{\omega}_{pb}^2 \left(1 - f\frac{r_b^2}{r_w^2}\right)$$

- Two-stream instability is strongest in limit of small axial momentum spreads

$$\left|\frac{\omega}{k_z} - V_b\right| \gg v_{Tzb}, \qquad \left|\frac{\omega}{k_z}\right| \gg v_{Tze}$$

Dipole-Mode Two-Stream Instability for $v_{Tzj} = 0$

- Setting $v_{Tzj} = 0$, for small values of f (and therefore ω_f^4), the kinetic dispersion relation supports four solutions with frequencies $\omega - k_z V_b \simeq \pm\omega_b$ and $\omega \simeq \pm\omega_e$.

- For $f \neq 0$, one of these solutions is unstable ($Im\omega > 0$). The unstable branch has real frequency and wavenumber (ω, k_z) closely tuned to (ω_0, k_{z0}) defined by

$$\omega_0 = \omega_e$$

$$\omega_0 - k_{z0} V_b = -\omega_b$$

- Expressing $\omega = \omega_0 + \delta\omega$ and $k_z = k_{z0} + \delta k_z$, the quartic dispersion relation can be approximated by the quadratic form

$$\delta\omega(\delta\omega - \delta k_z V_b) = -\frac{\omega_f^4}{4\omega_e\omega_b} \equiv -\Gamma_0^2$$

where $|\delta\omega| \ll 2\omega_e$ and $|\delta\omega - V_b\delta k_z| \ll 2\omega_b$ are assumed.

Dipole-Mode Two-Stream Instability for $v_{Tzj} = 0$

- Solving the (approximate) quadratic dispersion relation gives

$$
\begin{aligned}
Re\delta\omega &= \frac{1}{2}\delta k_z V_b \\
Im\delta\omega &= \Gamma_0[1 - (\delta k_z V_b/2\Gamma_0)^2]^{1/2}
\end{aligned}
$$

for the unstable branch with $Im\delta\omega > 0$.

- Validity of this result requires

$$\frac{1}{16}\omega_f^4 \ll \omega_b^3\omega_e \ , \quad \omega_b\omega_e^3$$

which is readily satisfied for $0 \leq f \leq 1$ and $\widehat{\omega}_{pb}^2/\omega_{\beta b}^2 \lesssim 0.5$.

- For $r_w/r_b \to \infty$, the maximum growth rate $(Im\delta\omega)_{max} = \Gamma_0$ is given by

$$\frac{(Im\delta\omega)_{max}}{\omega_{\beta b}} = \frac{1}{2^{7/4}} \frac{f^{1/2}(\gamma_b m_b/Z_b m_e)^{1/4}(\widehat{\omega}_{pb}^2/\omega_{\beta b}^2)^{3/4}}{[1 + (f/2)\widehat{\omega}_{pb}^2/\omega_{\beta b}^2]^{1/4}}$$

Dipole-Mode Two-Stream Instability for $v_{Tzj} = 0$

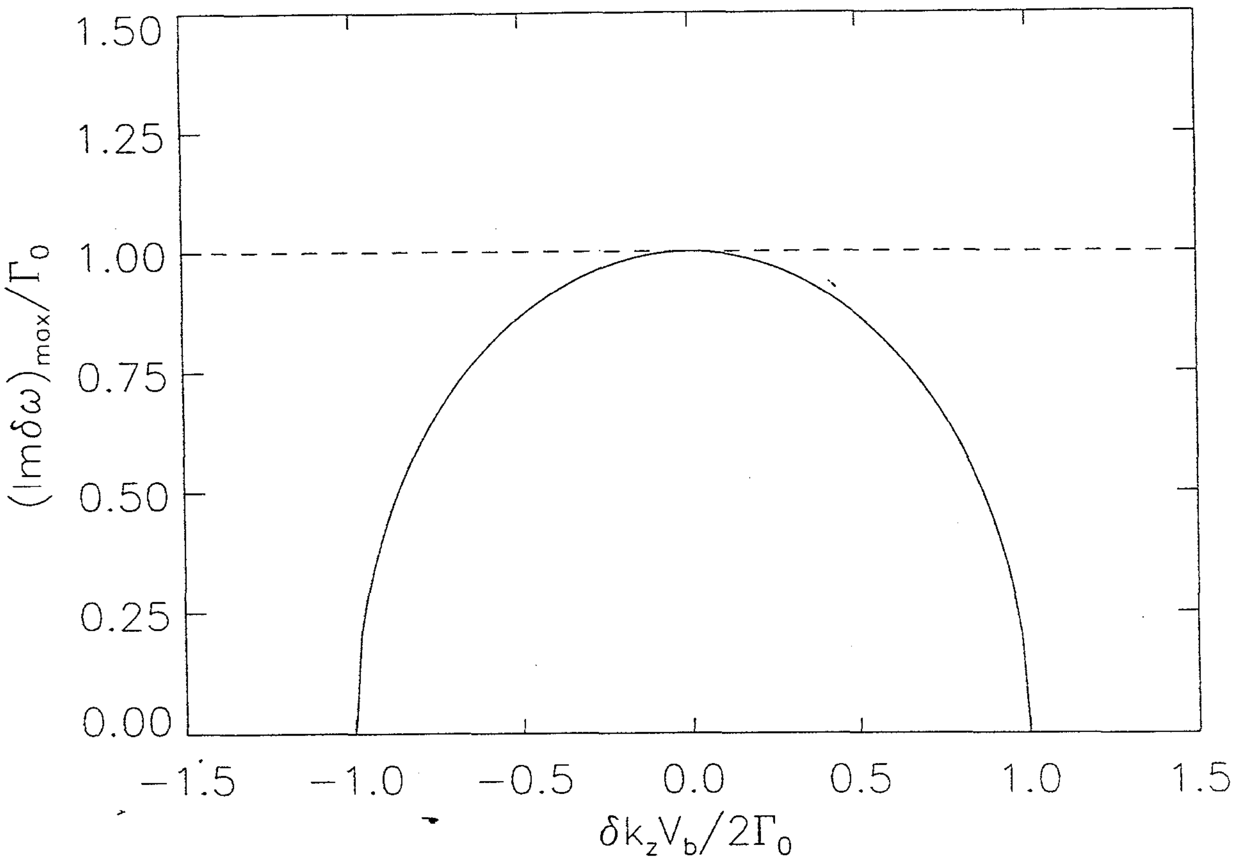
- For the case of step-function density profiles:
  - Growth rate increases with increasing beam intensity ($\hat{\omega}_{pb}^2$) and increasing fractional charge neutralization (f).
  - Growth rate decreases with increasing wall proximity (larger r_b/r_w).

- Growth rate can be substantial for high-intensity proton linacs and storage rings. For example, for a proton beam with $Z_b = 1$, $m_b/m_e = 1836$, $\gamma_b = 1.85$, $\hat{\omega}_{pb}^2/\omega_{\beta b}^2 = 0.1$, and $f = 0.1$, we find

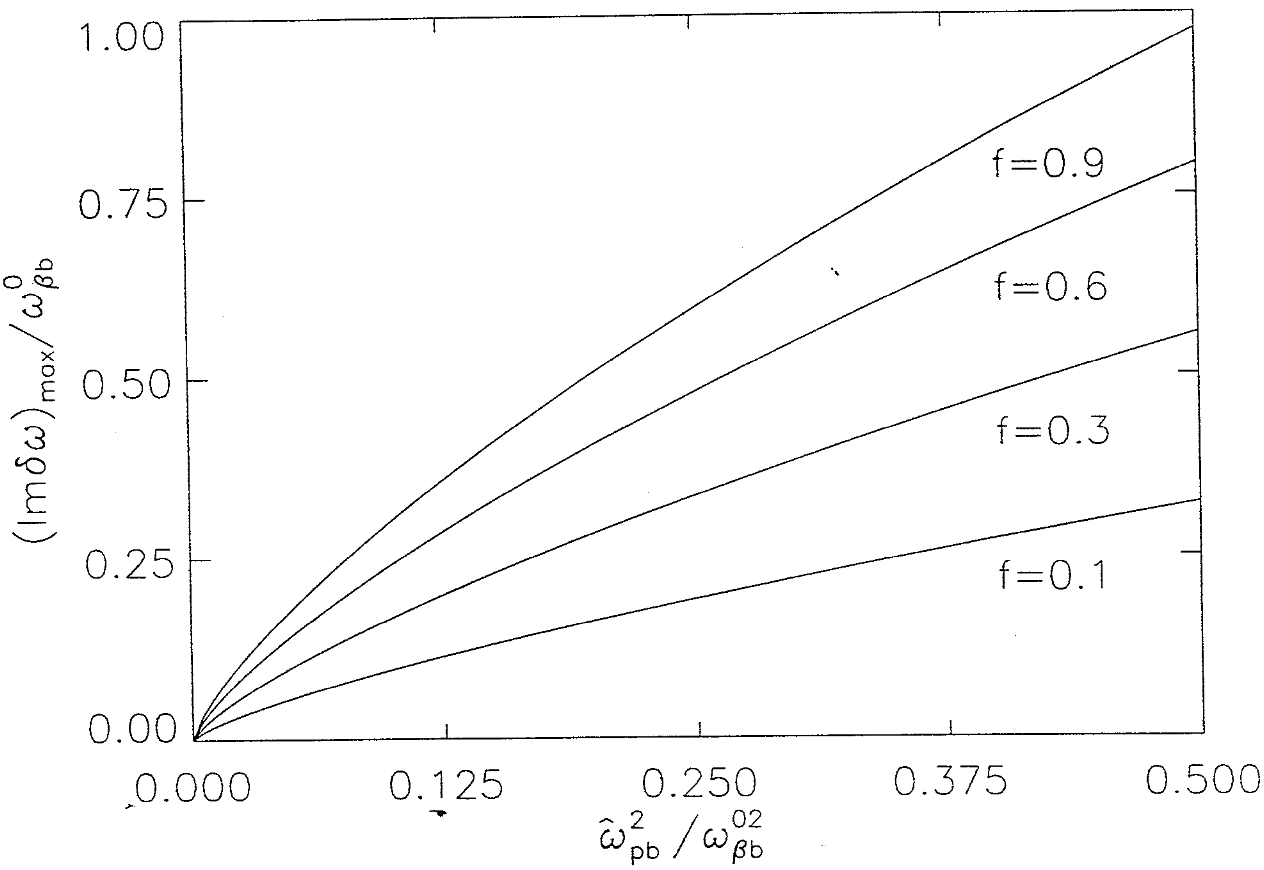
$$(Im\delta\omega)_{max} = 0.127\omega_{\beta b}$$

$$Re\omega \simeq \omega_0 = 13.03\omega_{\beta b}$$

$$k_z V_b \simeq k_{z0} V_b = 14.03\omega_{\beta b}$$



1.50
1.25
1.00
0.75
0.50
0.25
0.00
$(\mathrm{Im}\,\delta\omega)_{max}/\Gamma_0$
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
$\delta k_z V_b / 2\Gamma_0$



1.00
0.75
0.50
0.25
0.00
$(\text{Im}\,\delta\omega)_{max}/\omega^0_{\beta b}$
f=0.9
f=0.6
f=0.3
f=0.1
0.000
0.125
0.250
0.375
0.500
$\hat{\omega}^2_{pb}/\omega^{02}_{\beta b}$

Dipole-Mode Two-Stream Instability for $v_{Tzj} = 0$

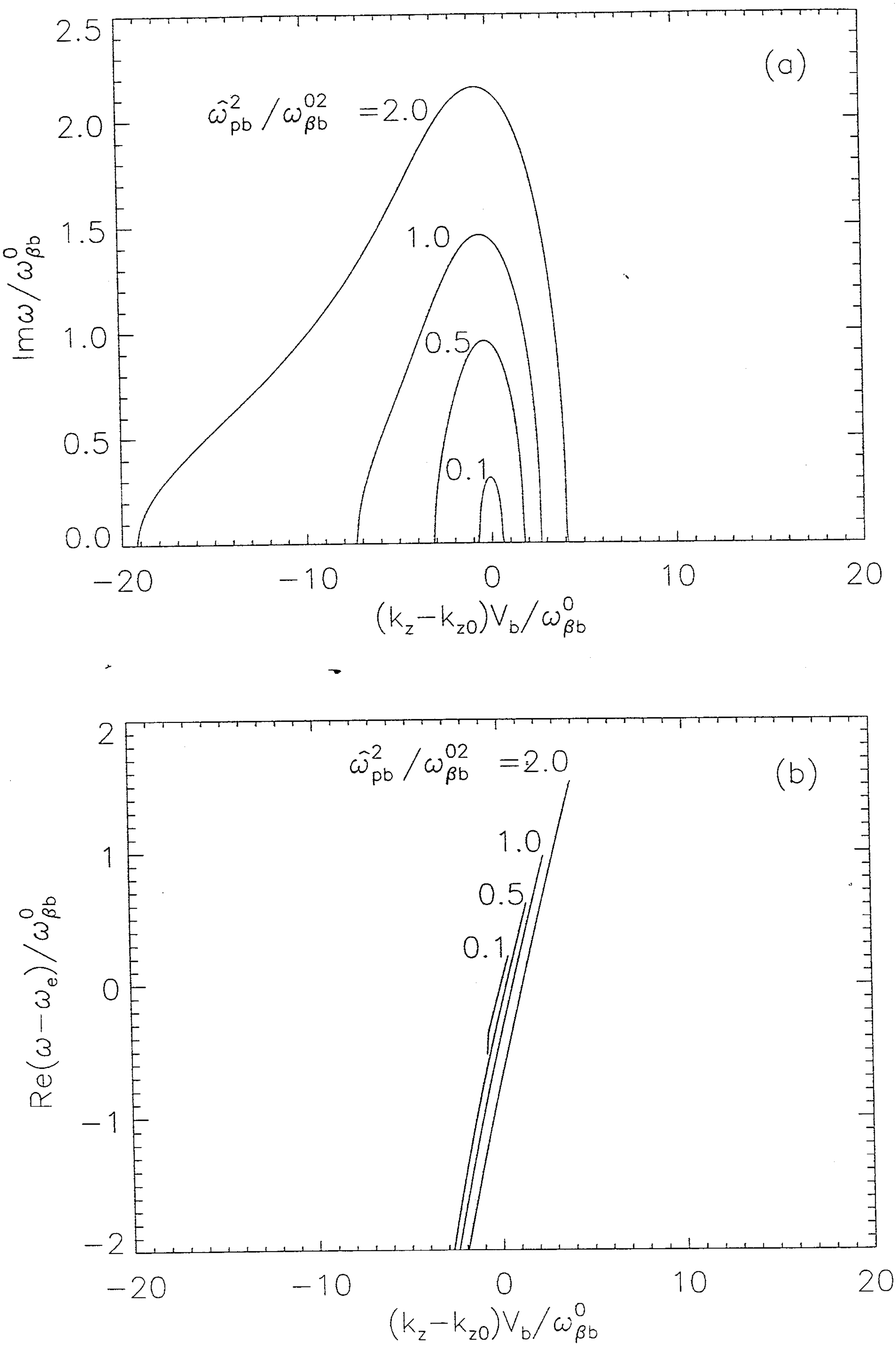
- For the ion beam parameters of interest for heavy ion fusion, the transverse beam emittance is small (small $\hat{T}_{\perp b}$), and the beam intensity is close to the space charge limit

$$\left(\frac{\hat{\omega}_{pb}^2}{\omega_{\beta b}^{02}}\right)_{max} = 2\gamma_b^2$$

for $f = 0$ and $2\hat{T}_{\perp b}/\gamma_b m_b \hat{\nu}_b^2 r_b^2 \ll 1$.

- At such high beam intensities, a cubic or full quartic approximation to the dispersion relation must be solved. For a heavy ion beam with $A = 133$ and $Z_b = 1$, kinetic energy $(\gamma_b - 1)m_b c^2 = 10$ GeV, ratio of beam radius to wall radius $r_b/r_w = 0.5$, and fractional charge neutralization $f = 0.1$, we obtain the maximum growth rate

$$(Im\omega)_{max} = 2.04\omega_{\beta b}$$



(a)
$\hat{\omega}_{pb}^2 / \omega_{\beta b}^{02} = 2.0$
1.0
0.5
0.1
$\mathrm{Im}\,\omega / \omega_{\beta b}^0$
$(k_z - k_{z0})V_b / \omega_{\beta b}^0$
(b)
$\hat{\omega}_{pb}^2 / \omega_{\beta b}^{02} = 2.0$
1.0
0.5
0.1
$\mathrm{Re}(\omega - \omega_e) / \omega_{\beta b}^0$
$(k_z - k_{z0})V_b / \omega_{\beta b}^0$

Effects of a Spread in Axial Momentum

- Incorporating an axial momentum spread, the kinetic dispersion relation for dipole-mode perturbations ($l = 1$) is given by

  $$[(\omega - k_z V_b + i|k_z|v_{Tbz})^2 - \omega_b^2][(\omega + i|k_z|v_{Tez})^2 - \omega_e^2] = \omega_f^4$$

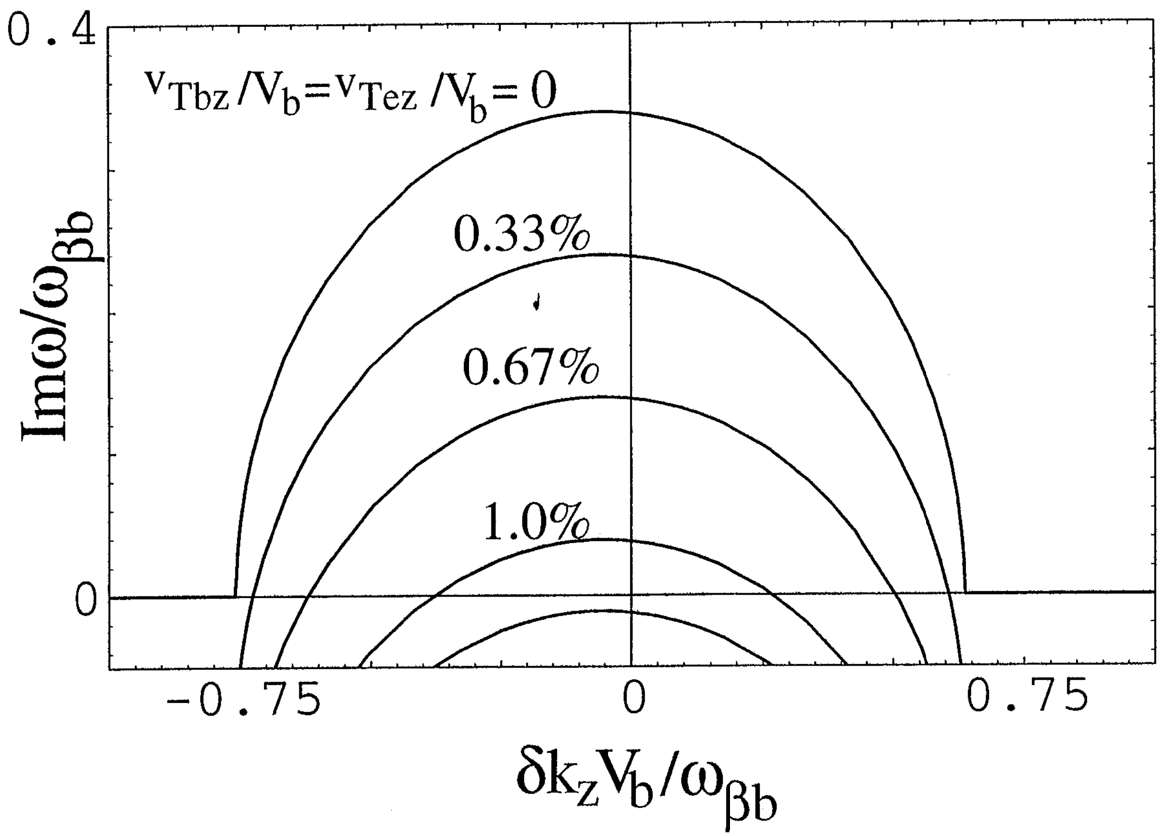
  where

  $$\omega_f^4 \equiv \frac{1}{4} f \left(1 - \frac{r_b^2}{r_w^2}\right)^2 \frac{\gamma_b m_b}{Z_b m_e} \widehat{\omega}_{pb}^4$$

  .

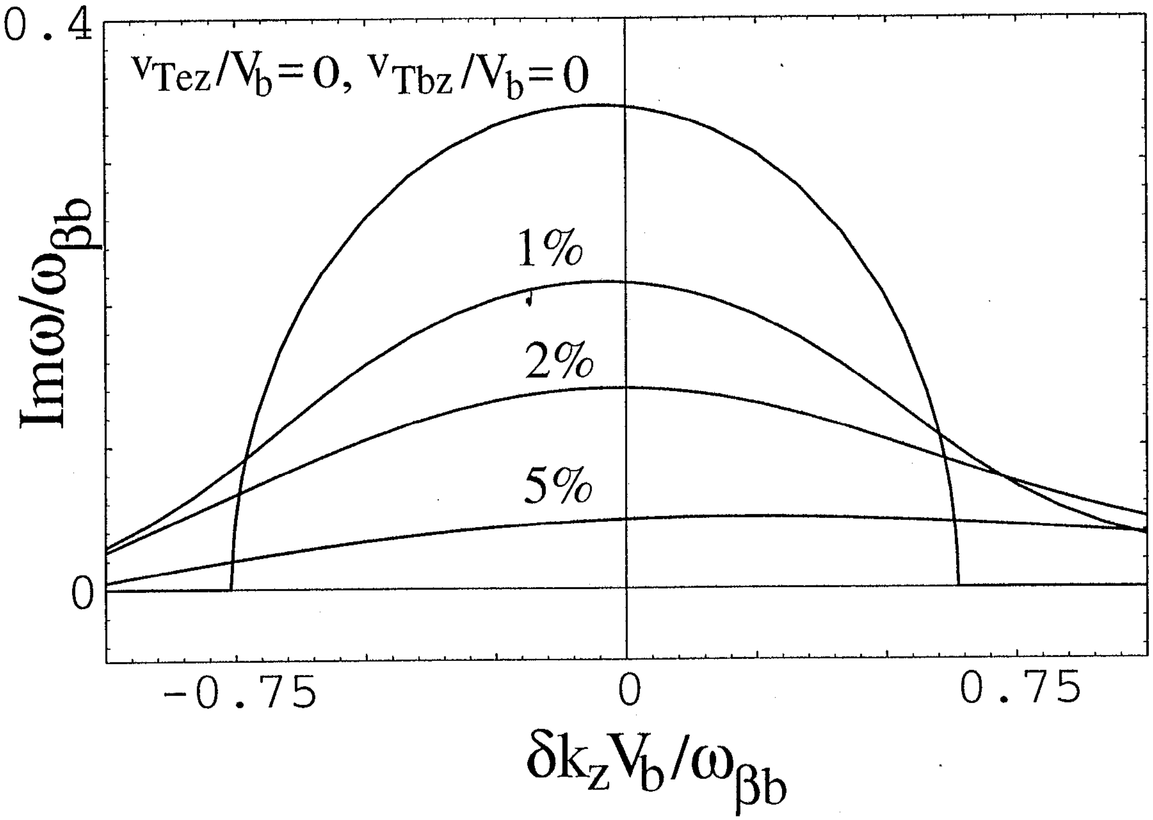
- Numerical analysis of this dispersion relation shows that a modest axial momentum spread can stabilize the dipole-mode two-stream instability at moderate values of beam intensity and fractional charge neutralization.

Effects of a Spread in Axial Momentum



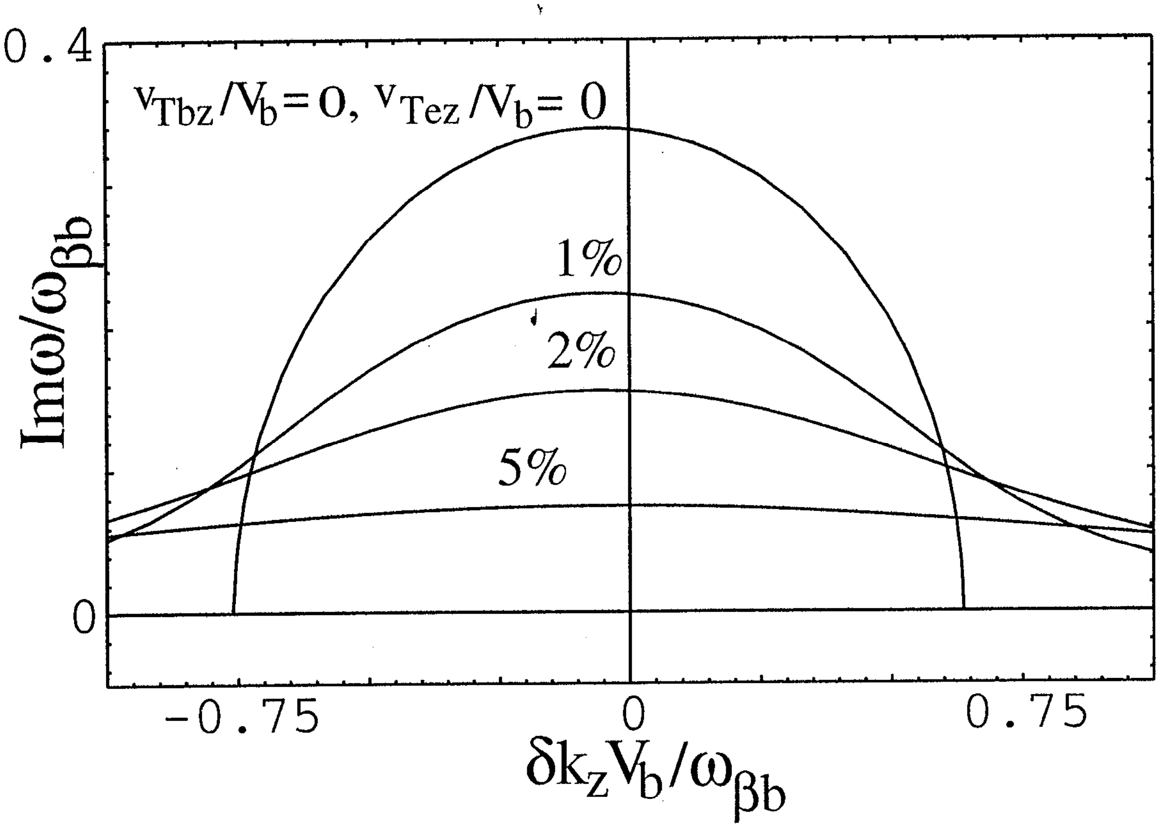
- $\gamma_b = 1.85$, $m_e/m_b = 1/1836$, $\hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta b}^2 = 0.079$,
- $v_{Tbz}/V_b = v_{Tez}/V_b = 0 \sim 1.0\%$.

Effects of a Spread in Axial Momentum

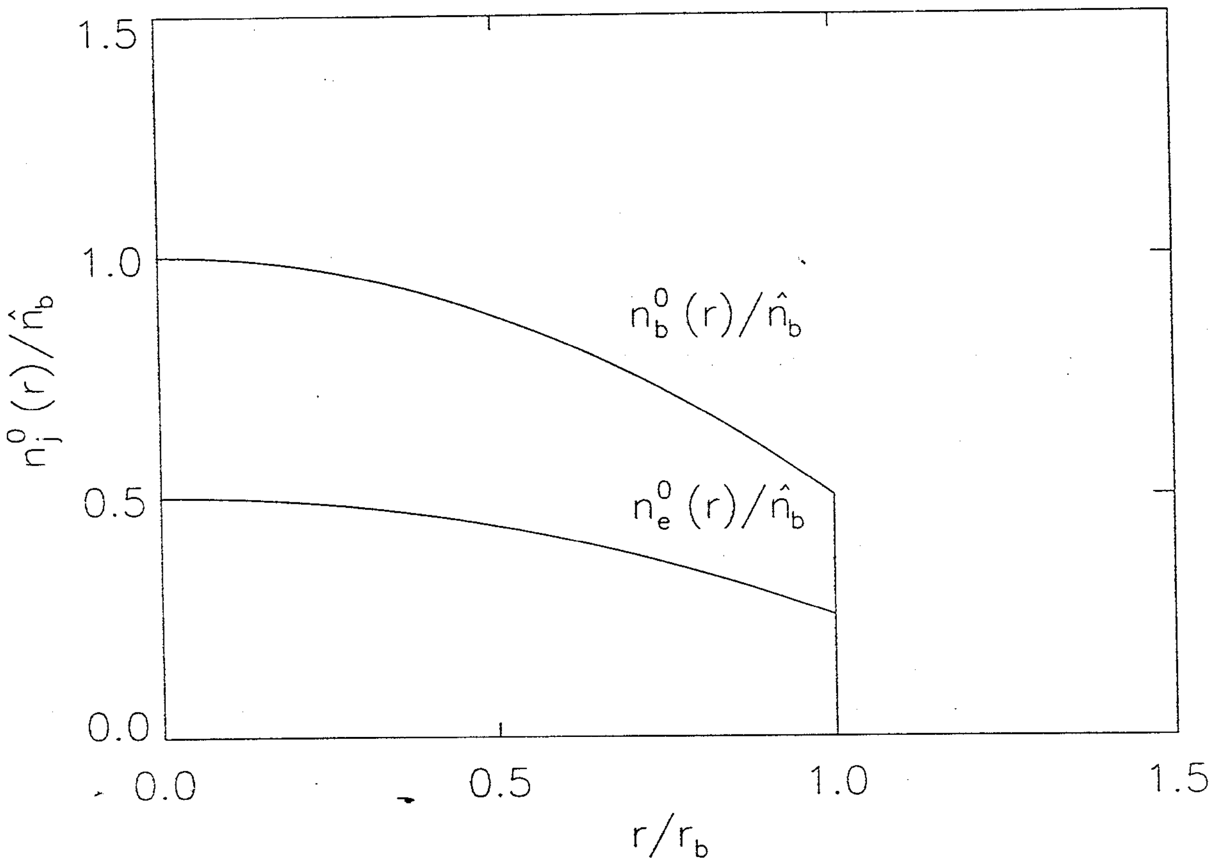


- $\gamma_b = 1.85$, $m_e/m_b = 1/1836$, $\hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta b}^2 = 0.079$
- $v_{Tez}/V_b = 0$, $v_{Tbz}/V_b = 0 \sim 5.0\%$.

Effects of a Spread in Axial Momentum



- $\gamma_b = 1.85$, $m_e/m_b = 1/1836$, $\hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta b}^2 = 0.079$
- $v_{Tbz}/V_b = 0$, $v_{Tez}/V_b = 0 \sim 5.0\%$.



1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
$n_j^0(r)/\hat{n}_b$
$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
r/r_b

Effects of a Spread in Transverse Betatron Frequencies for $v_{Tzj} = 0$

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- Assume (weak) parabolic variation in radial density profiles with

$$n_j^0(r) = \begin{cases} \hat{n}_j \left(1 - \epsilon \frac{r^2}{r_b^2}\right) , & 0 \le r < r_b \\ 0 , & r_b < r \le r_w \end{cases}$$

where $\epsilon \ll 1$.

- Adopt a model that makes the susceptibility replacements

$$\frac{\hat{\omega}_{pe}^2}{\omega^2 - \hat{\nu}_e^2} \to \frac{2}{r_b^2} \int_0^{r_b} \frac{dr r \omega_{pe}^2(r)}{\omega^2 - \nu_e^2(r)}$$

$$\frac{\hat{\omega}_{pb}^2}{(\omega - k_z V_b)^2 - \hat{\nu}_b^2} \to \frac{2}{r_b^2} \int_0^{r_b} \frac{dr r \omega_{pb}^2(r)}{(\omega - k_z V_b)^2 - \nu_b^2(r)}$$

Effects of a Spread in Transverse Betatron Frequencies for $v_{Tzj} = 0$

- For $\epsilon \ll 1$ and $\ell = 1$, dispersion relation becomes

$$[(\omega - k_z V_b)^2 - \omega_b^2 + \Delta\omega_b^2][\omega^2 - \omega_e^2 + \Delta\omega_e^2] = \omega_f^4 (1 - \epsilon\alpha_e)(1 - \epsilon\alpha_b) ,$$

where $\Delta\omega_e^2$ and $\Delta\omega_b^2$ are proportional to ϵ and related to the spreads in betatron frequencies.

- Threshold condition for the onset of instability is

$$\frac{f\widehat{\omega}_{pb}/\omega_{\beta b}}{(1 + f\widehat{\omega}_{pb}^2/2\omega_{\beta b}^2)^{1/2}} > \epsilon^2 \left(\frac{2\gamma_b m_b}{Z_b m_e}\right)^{1/2} \left(\frac{3}{8} + \frac{5}{8} f\right)^2$$

- As a simple example, for inhomogeneity parameter $\epsilon = 0.075$, and protons with $\gamma_b = 1.85$, $Z_b = 1$ and $m_b/m_e = 1836$, the threshold condition becomes

$$\frac{f\widehat{\omega}_{pb}/\omega_{\beta b}}{(1 + f\widehat{\omega}_{pb}^2/2\omega_{\beta b}^2)^{1/2}} > 0.065 \left(1 + \frac{5}{3} f\right)^2$$

Conclusions

- Using a fully kinetic model based on the Vlasov-Maxwell equation, we have derived the dispersion relation for the two-stream instability for a high-intensity ion beam propagating through a population of background electrons.

- The electron-ion two-stream instability is strongest (largest growth rate) for dipole-mode perturbations with azimuthal mode number $l = 1$.

- In the unstable regime, the two-stream instability growth rate is found to increase with increasing beam intensity, and increasing fractional charge neutralization.

- Effects that reduce the growth rate of the two-stream instability include:
  - Proximity of a conducting wall.
  - Axial momentum spread.
  - Spread in (depressed) transverse betatron frequencies.

Future Analytical and Numerical Studies of the Two-Stream Instability

- Determine mode structure for collective ion beam oscillations in the absence of electrons. Determine dependence on:
  - Beam intensity
  - Ion density profile shape
  - Spread in (depressed) transverse betatron frequency
  - Choice of input ion distribution function $f_b^0(r, \mathbf{p})$
- Determine properties of two-stream instability in presence of a small population of electrons at moderate ion beam intensity. Determine dependence of instability properties on:
  - Electron density profile shape
  - Spread in transverse electron betatron frequency
  - Ion beam intensity
  - Choice of input electron distribution function $f_e^0(r, \mathbf{p})$

Future Analytical and Numerical Studies of the Two-Stream Instability

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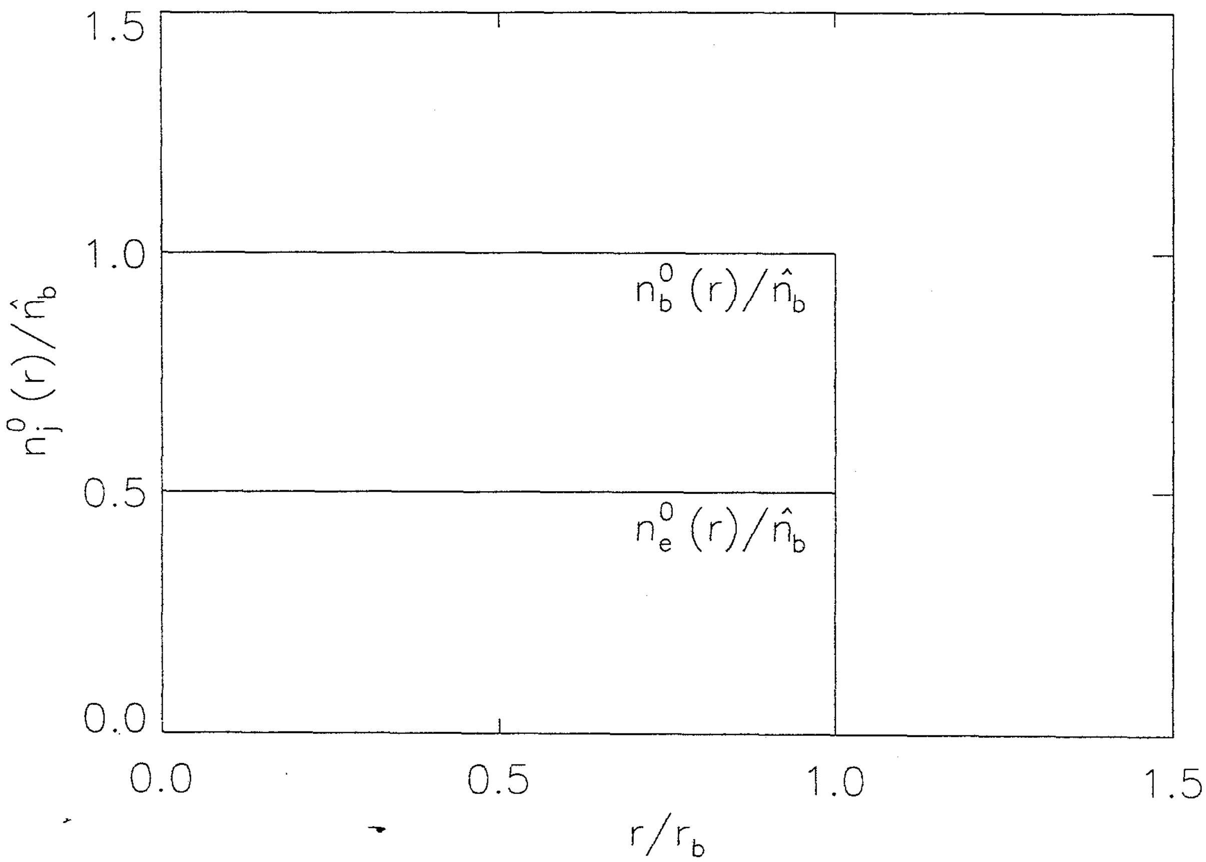
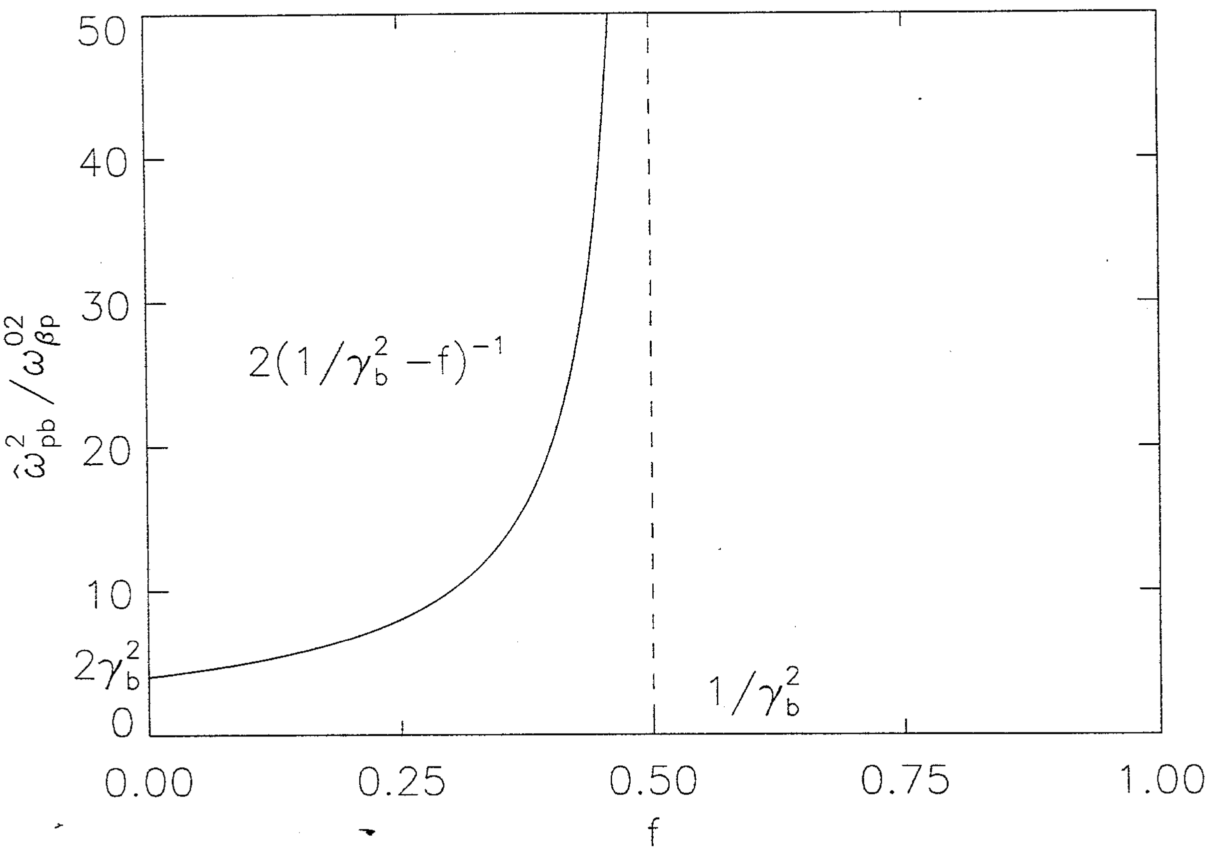
- Determine threshold properties of the e-p instability as a function of:
  - Beam intensity
  - Fractional charge neutralization
  - Choices of input distributions $f_b^0(r, \mathbf{p})$ and $f_e^0(r, \mathbf{p})$
  - Axial momentum spread

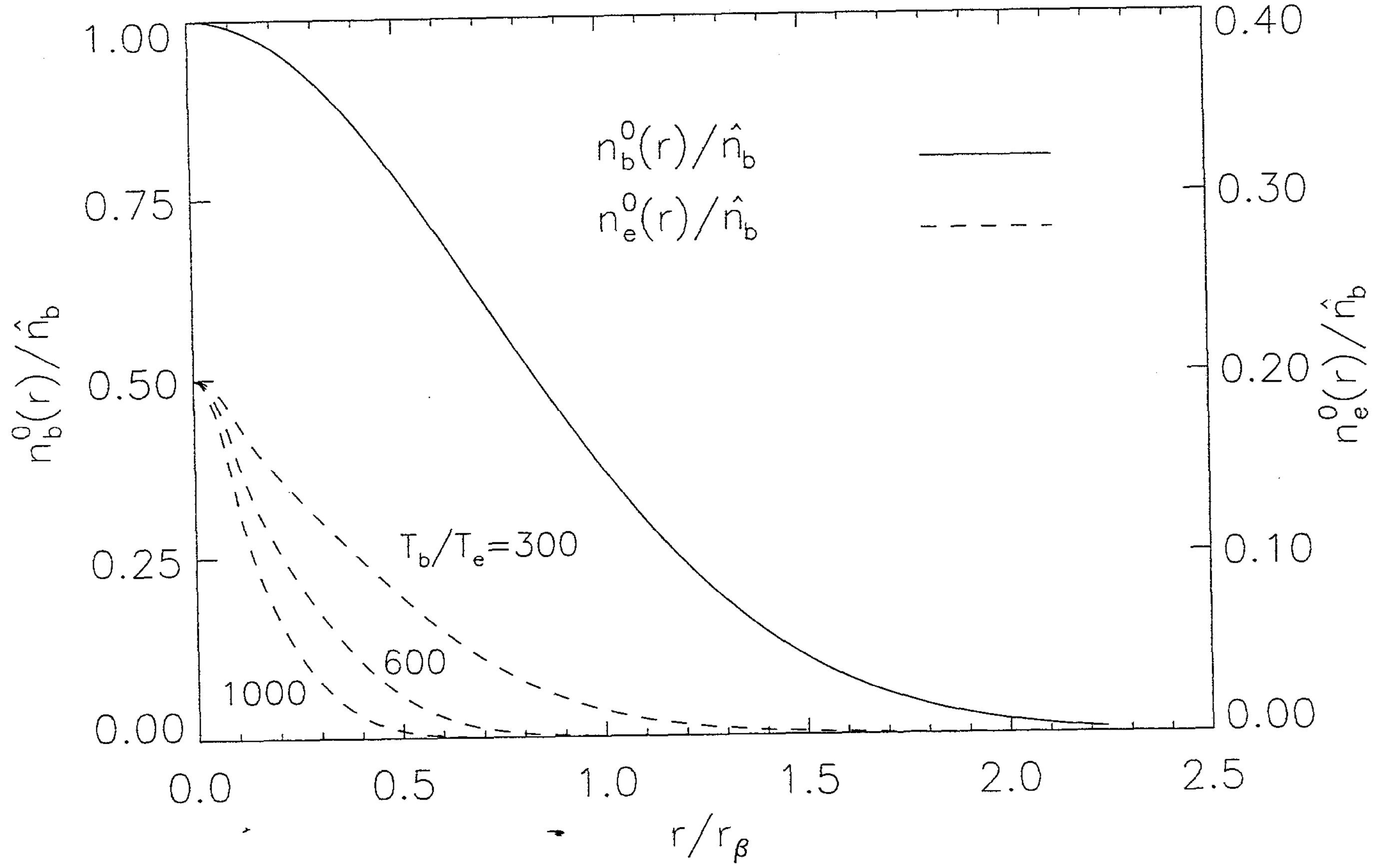
- Determine illustrative operating regimes for PSR and SNS that minimize the deleterious effects of the two-stream instability and maximize the threshold beam intensity for onset of the two-stream instability.



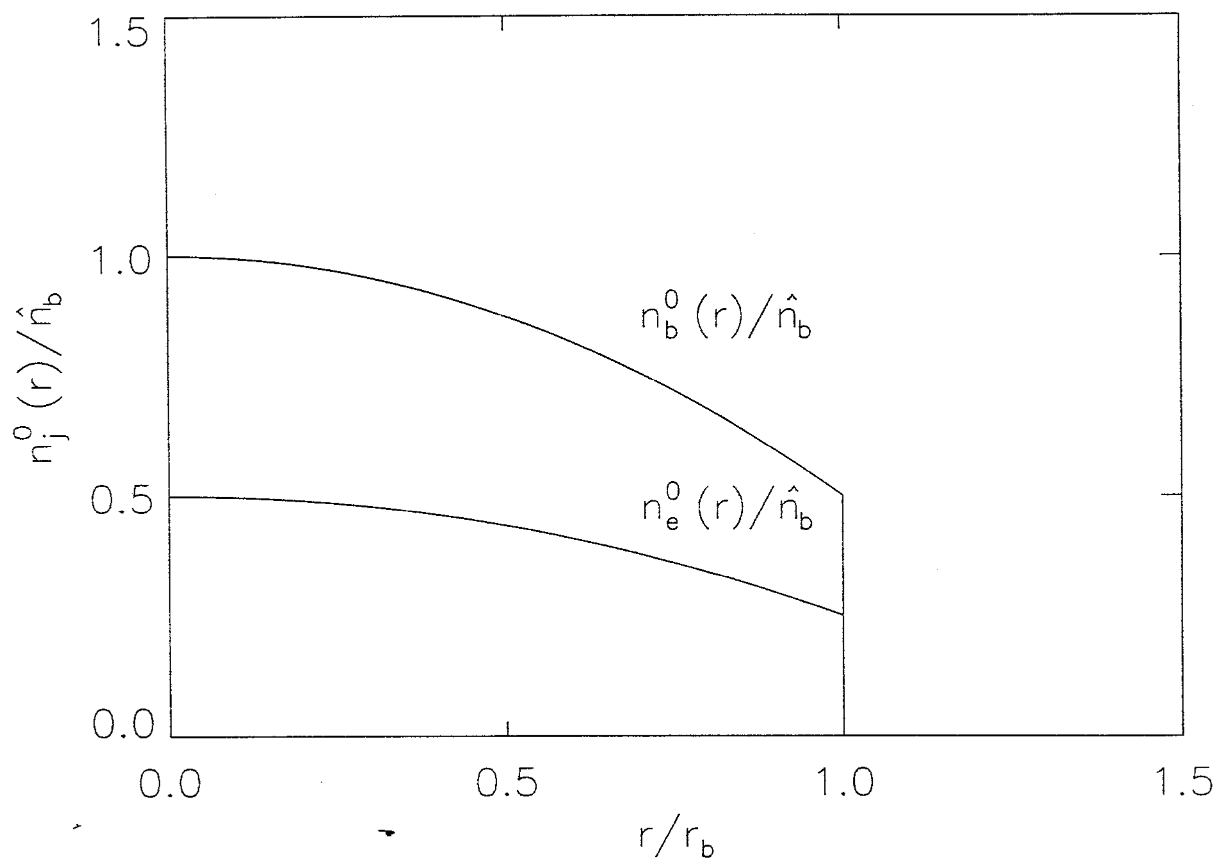
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
$n_j^0(r)/\hat{n}_b$
$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
r/r_b

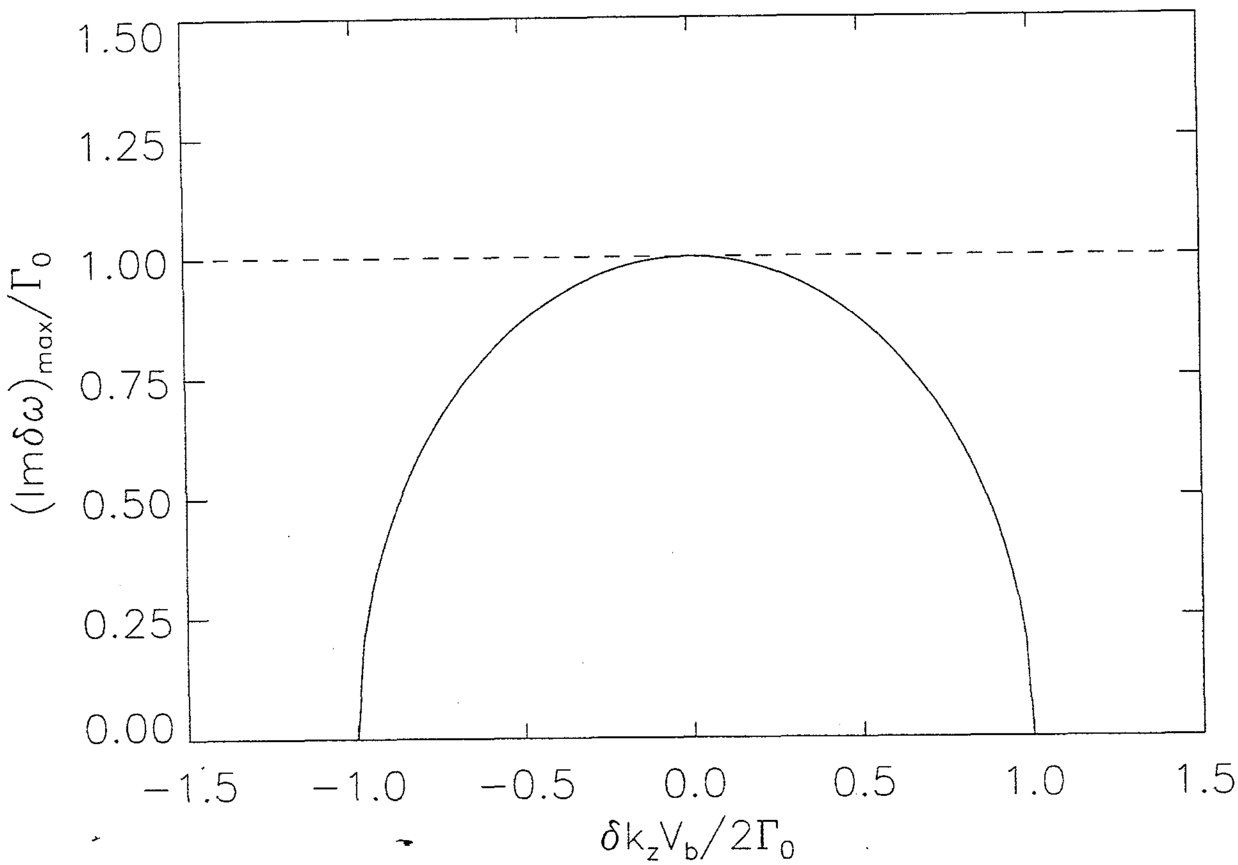


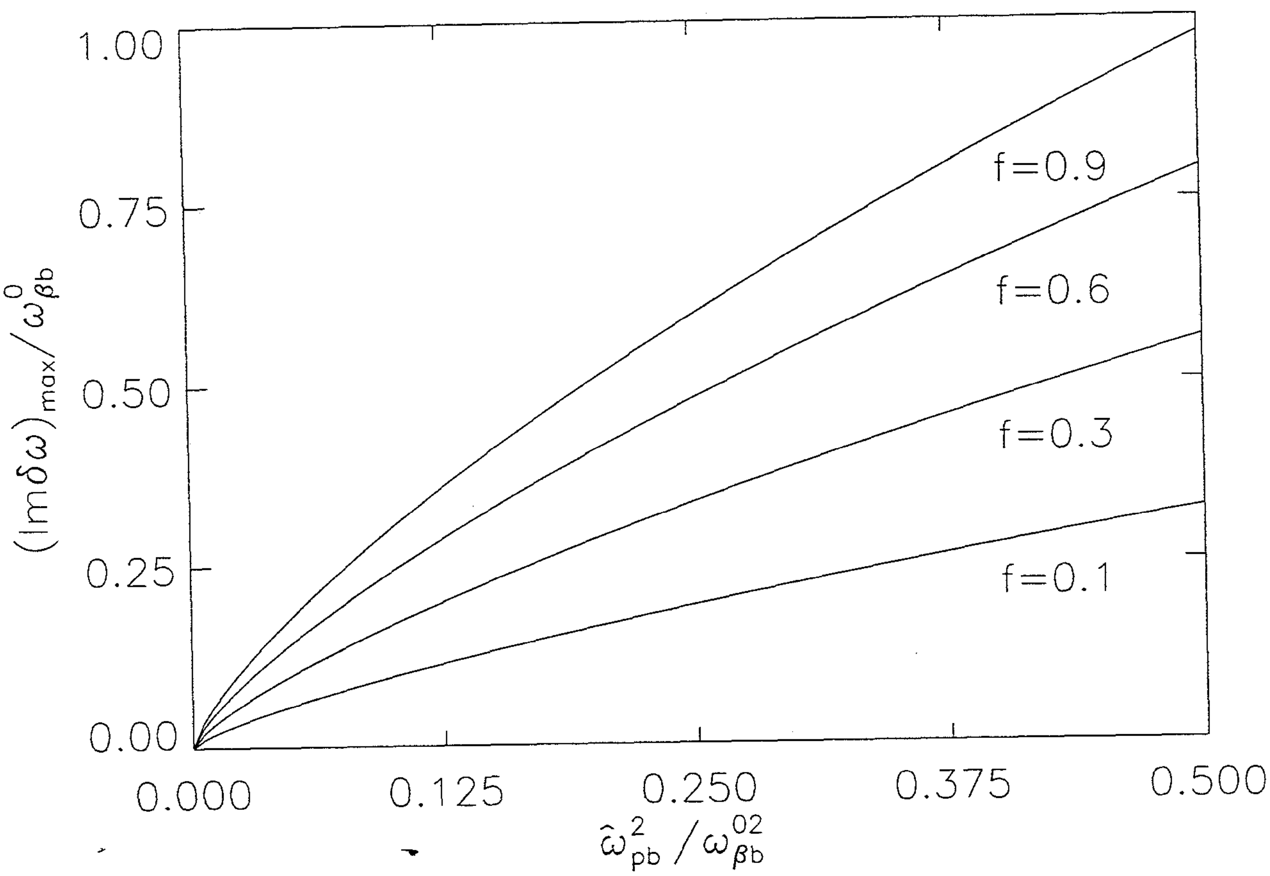
50
40
30
20
10
0
$2\gamma_b^2$
$\hat{\omega}_{pb}^2/\omega_{\beta p}^{02}$
$2(1/\gamma_b^2 - f)^{-1}$
$1/\gamma_b^2$
0.00
0.25
0.50
0.75
1.00
f

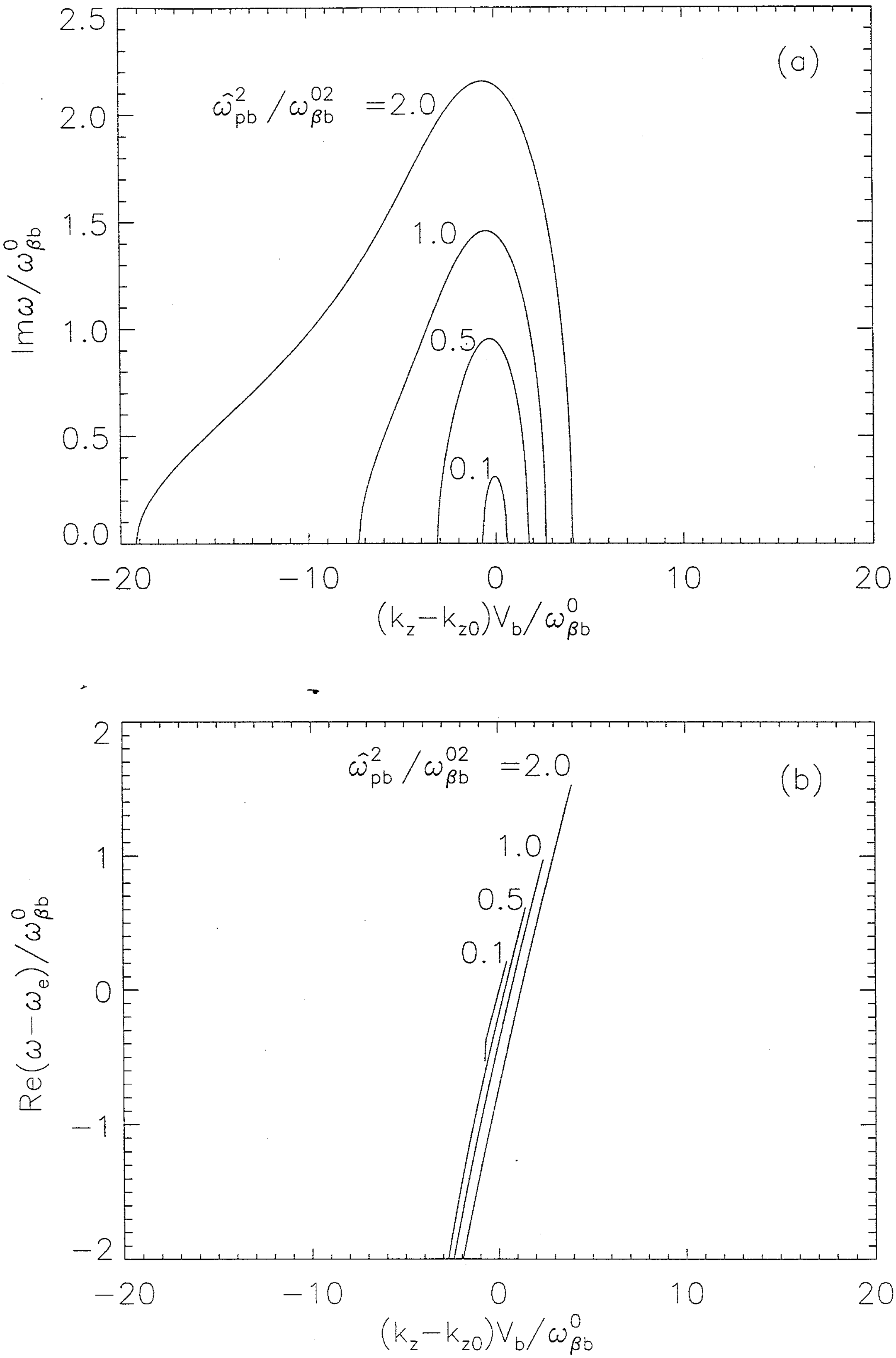


1.00
0.75
0.50
0.25
0.00
0.40
0.30
0.20
0.10
0.00
0.0
0.5
1.0
1.5
2.0
2.5
$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
$T_b/T_e=300$
600
1000
r/r_β


1.5
1.0
0.5
0.0
$n_j^0(r)/\hat{n}_b$
$n_b^0(r)/\hat{n}_b$
$n_e^0(r)/\hat{n}_b$
0.0
0.5
1.0
1.5
r/r_b


1.50
1.25
1.00
0.75
0.50
0.25
0.00
$(\mathrm{Im}\,\delta\omega)_{max}/\Gamma_0$
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
$\delta k_z V_b/2\Gamma_0$


1.00
0.75
0.50
0.25
0.00
0.000
0.125
0.250
0.375
0.500
f=0.9
f=0.6
f=0.3
f=0.1
$(\mathrm{Im}\,\delta\omega)_{max}/\omega^0_{\beta b}$
$\hat{\omega}^2_{pb}/\omega^{02}_{\beta b}$



(a)
$\hat{\omega}_{pb}^{2}/\omega_{\beta b}^{02}$ =2.0
1.0
0.5
0.1
$\mathrm{Im}\,\omega/\omega_{\beta b}^{0}$
$(k_z-k_{z0})V_b/\omega_{\beta b}^{0}$
0.0
0.5
1.0
1.5
2.0
2.5
−20
−10
0
10
20
(b)
$\hat{\omega}_{pb}^{2}/\omega_{\beta b}^{02}$ =2.0
1.0
0.5
0.1
$\mathrm{Re}(\omega-\omega_e)/\omega_{\beta b}^{0}$
$(k_z-k_{z0})V_b/\omega_{\beta b}^{0}$
−2
−1
0
1
2
−20
−10
0
10
20