17TH ADVANCED BEAM DYNAMICS WORKSHOP ON
FUTURE LIGHT SOURCES

Several Formulas for High Gain FEL

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1. Scaling gain function

(Yu, Krinsky, Gluckstern, PRL (1990)) 64 3011

$$\frac{1}{L_G} = k_w D \; G\left(k_s \epsilon, \frac{\sigma_\gamma}{\gamma D}, \frac{k_\beta}{k_w D}, \frac{K}{\gamma}\right)$$

1.1) Natural focusing and strong focusing

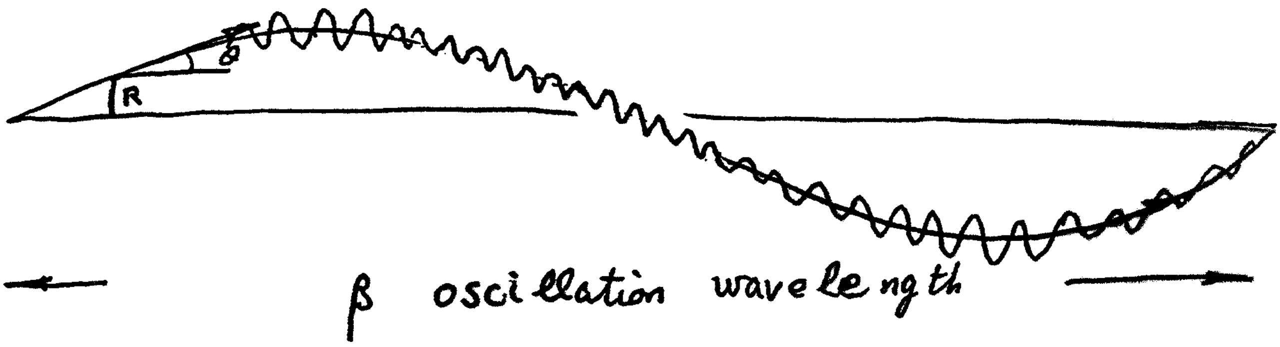
natural focusing

$$k_{\beta n} = \frac{K}{2\gamma} k_w$$

strong focusing

$$k_\beta > k_{\beta n}$$

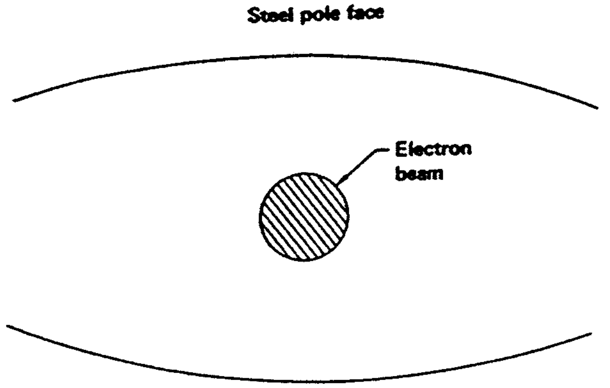
Focusing and Betatron Oscillation



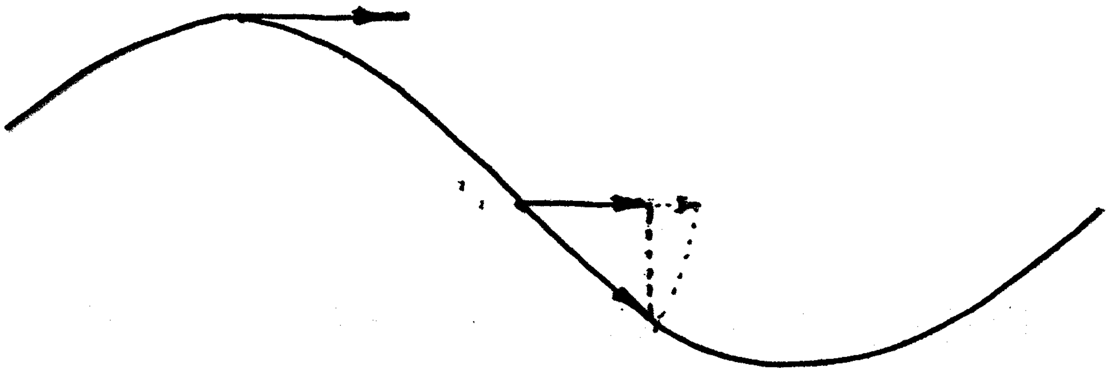
Natural focusing :

provided by field gradient in wiggl

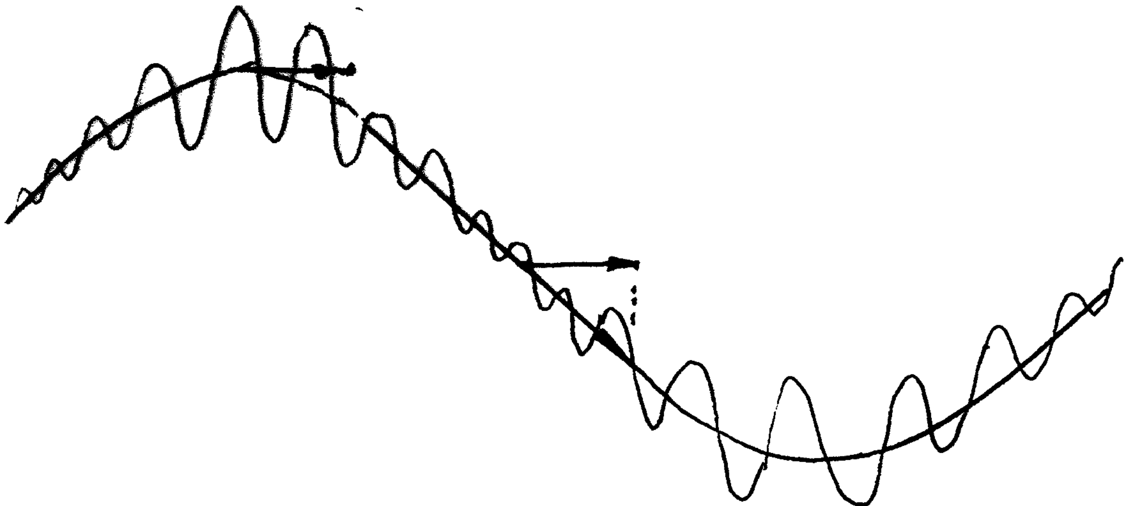
$$B_w = B_{w0}\left(1 + \frac{1}{4}\left(k_w^2 x^2 + k_w^2 y^2\right)\right) \cos k_w z$$



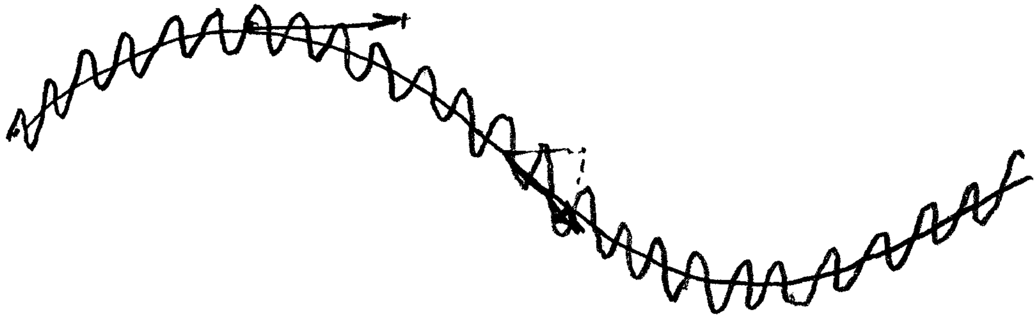
properly shaped
magnetic poles
can provide focusing
on both planes
(E.T. Scharlemann)



β-oscillation causes longitudinal velocity modulation



in natural focusing, the modulation is canceled

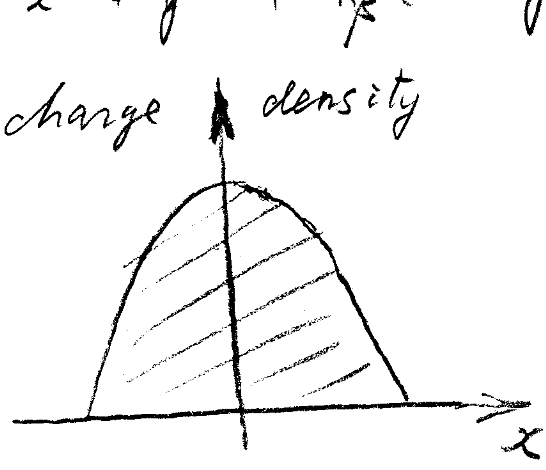


$$r = \frac{\frac{K}{2\gamma D}}{\frac{k_\beta}{k_w D}} = \frac{k_{\beta n}}{k_\beta} < 1$$

strong focusing : $r \ll 1$

1.2) Waterbag model

$$x'^2 + y'^2 + k_\beta^2 (x^2 + y^2) \le k_\beta^2 R^2$$



(*

$$P \propto e^2 \; \mathrm{Im}(\mu) \; k_w \; L_w$$

$$\frac{\mathrm{Im}(\mu)}{D} = x \;\; f[x,y,z] \;\; (1-r^2) + x \;\; g[x,y,z] \;\; r^2$$

$$D = \sqrt{\frac{8 I_0}{I_A} \;\; \frac{K^2}{\left(1+\frac{K^2}{2}\right)} \;\; \frac{J^2}{\gamma}}$$

$$x = \left(\frac{k_\beta}{k_w D} \cdot \frac{1}{12 \;\; k_s \, \epsilon}\right)^{\frac{1}{3}}$$

$$y = \frac{\sigma_\gamma}{\gamma \; D \; x}$$

$$z = \left(\frac{k_\beta}{k_w D} \cdot (12 \;\; k_s \, \epsilon)^2\right)^{\frac{2}{3}}$$

$$r = \frac{\frac{K}{2 \gamma D}}{\frac{k_\beta}{k_w D}}$$

*)

(* Physical meaning:

$$x=\frac{1}{\tilde{a}}=\frac{1}{\sqrt{2\sqrt{3}}}\sqrt{\frac{L^{1D}{}_G}{L_R}}$$ (diffraction)

$$y=\frac{\sigma_\gamma}{2\rho\gamma}$$ (energy spread)

$$z=12\sqrt{3}\,(k_\beta\ L_G{}^{1D})\ k_s\,\epsilon$$ (angular spread)

$$r=\frac{k_{\beta n}}{k_\beta}$$ (focusing type)

$$L_R=\frac{4\pi\sigma_x{}^2}{\lambda_s}$$ (Rayleigh Range. model indep.)

$$\epsilon=k_\beta\,\sigma_x{}^2$$ (emittance. model indep.)

$$L_G{}^{1D}=\frac{\lambda_w}{4\pi\sqrt{3}\,\rho}$$ (1 D gain length. model dep.

$$\rho=\left(\frac{D^2}{96\,k_s\,k_w\,\sigma_x{}^2}\right)^{\frac{1}{3}}$$ (Pierce parameter. model dep

*)

In[12]:= **Remove@@Names["@"]**

$$f[x\_,\ y\_,\ z\_]\ := \frac{\sqrt{3}}{2}\ \frac{f_1[x,\ z]}{f_2\,[x,\ z]}\ e^{f_3[x,z]\,y^2};$$

$f_1[x\_,\ z\_]\ := 1 + a_1\,x + a_2\,z + a_3\,x^2 + a_4\,z^2 + a_5\,x\,z;$

$f_2\,[x\_,\ z\_]\ := 1 + b_1\,x + b_2\,z + b_3\,x\,z + b_4\,x^2 + b_5\,z^2 + b_6\,x^2\,z + b_7\,x\,z^2 + b_8\,x^3 + b_9\,z^3;$

$f_3[x\_,\ z\_]\ := c_0 + c_1\,x + c_2\,z + c_3\,x\,z + c_4\,x^2 + c_5\,z^2 + c_6\,x^2\,z + c_7\,x\,z^2 + c_8\,x^3 + c_9\,z^3;$

$a_1 = 0.2284;\ a_2 = -0.1165;\ a_3 = -0.078;\ a_4 = 0.003616;\ a_5 = 0.006525;$
$b_1 = 0.6431;\ b_2 = -0.07293;\ b_3 = -0.1255;\ b_4 = 0.4192;$
$b_5 = 0.0003568;\ b_6 = 0.04701;\ b_7 = 0.004425;\ b_8 = -0.149;\ b_9 = 7.723\times 10^{-5};$
$c_0 = -2.041;\ c_1 = -9.154;\ c_2 = 0.08928;\ c_3 = 0.5068;\ c_4 = -1.806;$
$c_5 = -0.0001643;\ c_6 = -0.6042;\ c_7 = -0.04656;\ c_8 = -2.13;\ c_9 = 0.0001664;$

$$g[x\_,\ y\_,\ z\_]\ := \frac{\sqrt{3}}{2}\ \frac{g_1[x,\ z]}{g_2[x,\ z]}\ e^{g_3[x,z]\,y^2}\ g_4[x,\ y,\ z];$$

$g_1[x\_,\ z\_]\ := 1 + A_1\,x + A_2\,z + A_3\,x^2 + A_4\,z^2 + A_5\,x\,z;$

$g_2[x\_,\ z\_]\ := 1 + B_1\,x + B_2\,z + B_3\,x\,z + B_4\,x^2 + B_5\,z^2 + B_6\,x^2\,z + B_7\,x\,z^2 + B_8\,x^3 + B_9\,z^3;$

$g_3[x\_,\ z\_]\ := C_0 + C_1\,x + C_2\,z + C_3\,x\,z + C_4\,x^2 + C_5\,z^2 + C_6\,x^2\,z + C_7\,x\,z^2 + C_8\,x^3 + C_9\,z^3;$

$$g_4[x\_,\ y\_,\ z\_]\ := \frac{1 + (\alpha_1 + \alpha_2\,\frac{z}{x})\,y + (\alpha_3 + \alpha_4\,\frac{z}{x})\,y^2}{1 + (\beta_1 + \beta_2\,\frac{z}{x})\,y + (\beta_3 + \beta_4\,\frac{z}{x})\,y^2};$$

$A_1 = 0.4237;\ A_2 = -0.102;\ A_3 = -0.09466;\ A_4 = 0.004205;\ A_5 = -0.01283;$
$B_1 = 0.7361;\ B_2 = -0.04635;\ B_3 = -0.1577;\ B_4 = 0.6479;$
$B_5 = 0.001515;\ B_6 = 0.1261;\ B_7 = 0.005783;\ B_8 = -0.1909;\ B_9 = 7.478\times 10^{-5};$
$C_0 = -1.897;\ C_1 = -11.16;\ C_2 = -0.08763;\ C_3 = 3.041;\ C_4 = 0.05795;$
$C_5 = -0.003451;\ C_6 = -3.089;\ C_7 = -0.2007;\ C_8 = -2.065;\ C_9 = 0.001382;$
$\alpha_1 = 7.649;\ \alpha_2 = -0.08598;\ \alpha_3 = -10.25;\ \alpha_4 = 0.1067;$
$\beta_1 = 8.012;\ \beta_2 = -0.08598;\ \beta_3 = -11.22;\ \beta_4 = 0.1106;$

$$(*\ \ \frac{\mathrm{Im}\ (\mu)}{D} = G\ \left(\ k_s\,\epsilon,\ \frac{\sigma_\gamma}{\gamma\ D},\ \ \frac{k_\beta}{k_w\,D},\ \frac{k_{\beta n}}{k_\beta}\right)\ *)$$

imdw[ksep_, sigeD_, kbD_, knkb_] :=

$$\mathrm{Block}\Big[\{x,\ y,\ z\},\ x = \left(\mathrm{kbD}\ \ \frac{1}{12\ \mathrm{ksep}}\right)^{\frac{1}{3}};\ y = \frac{\mathrm{sigeD}}{x};\ z = (\mathrm{kbD}\ \ (12\ \mathrm{ksep})^2)^{\frac{2}{3}};$$

```
  r = knkb;
If[ x < 0.1 || z > 14 || x > 2 ||
    ( f[x, y, z] (1 - r^2) + g[x, y, z] r^2 ) < 0.1, Print["warning! x=", x,
    "  y=", y, "  z=", z, " Imλ=", f[x, y, z] (1 - r^2) + g[x, y, z] r^2]];
N[  x f[x, y, z] (1 - r^2) + x g[x, y, z] r^2 ] ]
```

2. Initialize FEL paramters

```
λβ = 101.56;
γ = 27978;
ϵn = 1.5×10^-6;
K = 3.6954;
IA = 17045;
I0 = 3400;
λw  = 0.03;
λs = 0.15×10^-9;
σγ = 2×10^-4;
```

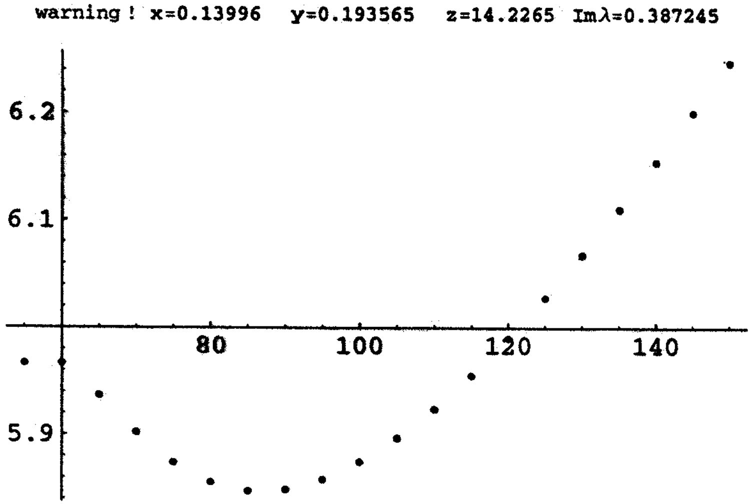
$\lambda_\beta = 101.56;$
$\gamma = 27978;$
$\epsilon_n = 1.5\times 10^{-6};$
$K = 3.6954;$
$I_A = 17045;$
$I_0 = 3400;$
$\lambda_w = 0.03;$
$\lambda_s = 0.15\times 10^{-9};$
$\sigma_\gamma = 2\times 10^{-4};$

```
In[17]:= waterfit = Table[
   {λ_β, Module[{}, ξ = K^2 /4/ (1+K^2 /2);
  J = BesselJ[0, ξ] - BesselJ[1, ξ];
  k_s = (2 π) / λ_s;
       ep = e_n / γ;
       ksep = k_s  ep;
  D_0 = √ ((8 I_0) / I_A
       K^2 / (1+K^2 /2)    J^2/γ);
  k_β  = (2 π) / λ_β;
   k_w = (2 π) / λ_w;
       koD = k_β / (k_w D_0);
       sigoD = σ_γ / D_0;
```

$$k_\beta = \frac{2\pi}{\lambda_\beta};$$

$$k_{\beta n} = \frac{K}{2\gamma}\frac{2\pi}{\lambda_w};$$

```
  L_G = λ_w / (4 π D_0
        imdw[ksep, sigoD, koD, k_βn / k_β]) ]}
  , {λ_β, 55, 150, 5}];
  ListPlot[waterfit]
```


warning ! x=0.13996 y=0.193565 z=14.2265 Imλ=0.387245
6.2
6.1
5.9
80
100
120
140

Different scaling functions

1. Existence of scaling relation

   Natural focusing, waterbag model

   (1990, Yu, Krinsky, Gluckstern)
   PRL, 64, 3011

2. Natural focusing, gaussian model

   Fit of scaling function

   (1992, Chin, Kim, Xie)
   PRA 46 6662

3. Strong focusing, waterbag

   (1994, Yu, Hung, Li, Krinsky)
   PRE, 51 813

4. Natural focusing, gaussian, another fit

   (1995, Xie, IEEE p.183)

1.3) Compare with gaussian model

$$\rho_g = \left(\frac{D^2}{64 k_s k_w \sigma_x^2} \right)^{\frac{1}{3}} = \left(\frac{3}{2}\right)^{\frac{1}{3}} \rho_w$$

$$\eta_d = \frac{1}{\sqrt{3}} \left(\frac{k_\beta}{k_w D} \cdot \frac{1}{k_s \varepsilon} \right)^{\frac{2}{3}} = \frac{L_{Gg}^{1D}}{L_R}$$

$$\eta_\gamma = \frac{4}{\sqrt{3}} \frac{\sigma_\gamma}{\gamma D} \left(\frac{k_\beta}{k_w D} \cdot \frac{1}{k_s \varepsilon} \right)^{-\frac{1}{3}} = \frac{1}{\sqrt{3}} \frac{\sigma_\gamma}{\gamma \rho_g}$$

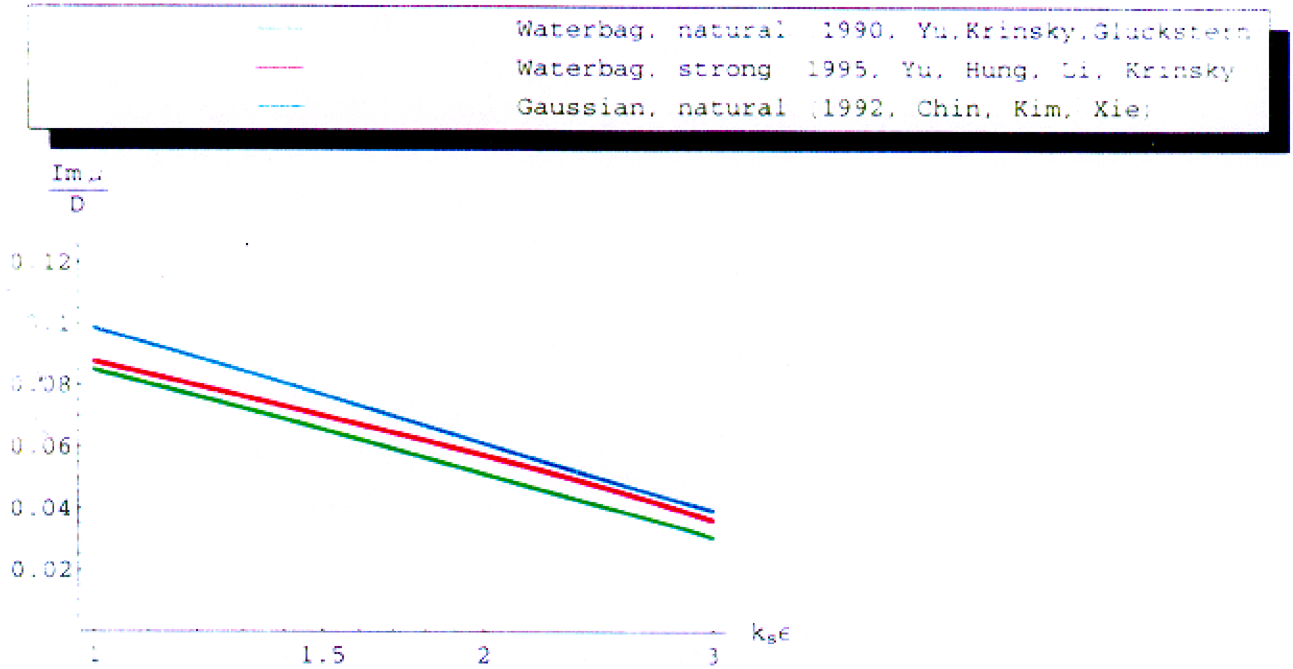
$$\eta_\varepsilon = \frac{4}{\sqrt{3}} \left(\frac{k_\beta}{k_w D} (k_s \varepsilon)^2 \right)^{\frac{2}{3}} = 2 \left(k_\beta L_{Gg}^{1D} \right) k_s \varepsilon$$

$$\frac{\mathrm{Im}(\mu)}{D} = \frac{\sqrt{3}}{2} \frac{1}{1+\eta} \cdot \frac{1}{2} \left(\frac{k_\beta}{k_w D} \cdot \frac{1}{k_s \varepsilon} \right)^{\frac{1}{3}}$$

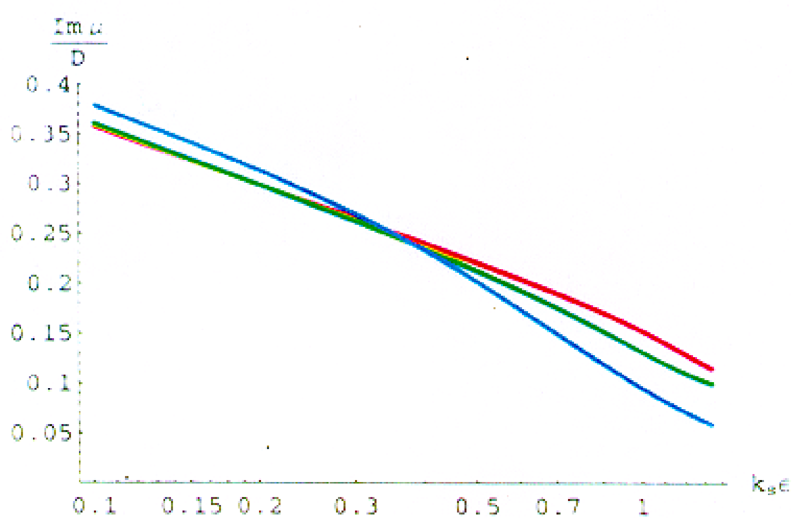
$$\eta = a_1 \eta_d^{a_2} + a_3 \eta_\varepsilon^{a_4} + a_5 \eta_\gamma^{a_6} + a_7 \eta_\varepsilon^{a_8} \eta_\gamma^{a_9}$$
$$+ a_{10} \eta_d^{a_{11}} \eta_\gamma^{a_{12}} + a_{13} \eta_d^{a_{14}} \eta_\varepsilon^{a_{15}} + a_{16} \eta_d^{a_{17}} \eta_\varepsilon^{a_{18}} \eta_\gamma^{a_{19}}$$

monotonious function of η_d, η_ε, η_γ

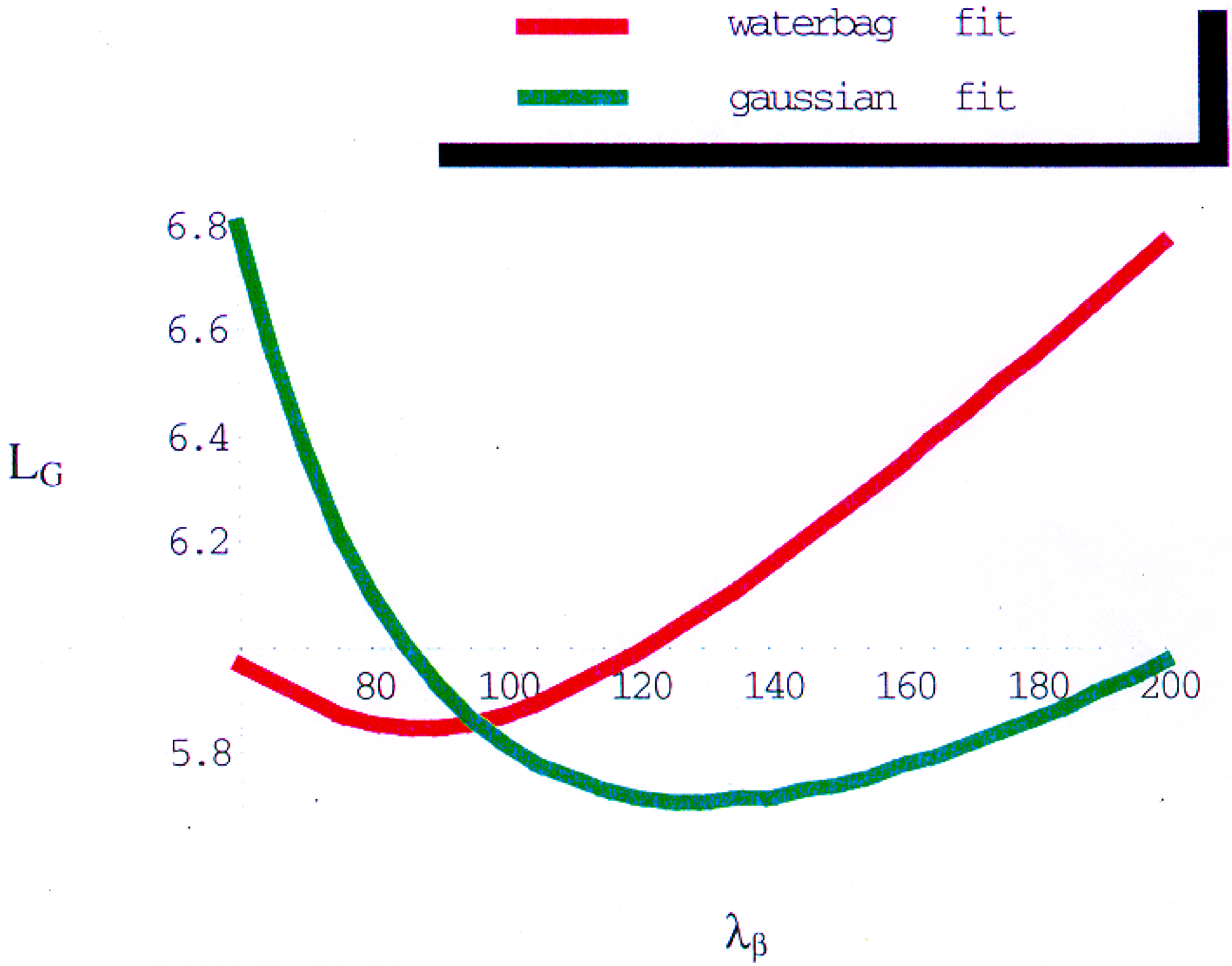
$$\frac{k_\beta}{k_\omega\, D} = 0.04$$



$$\frac{k_\beta}{k_\omega\, D} = 0.2$$



LCLS Gain Length
Compare waterbag with gaussian model



2. Effect of Wiggler Errors

(Yu, Krinsky, Gluckstern, van Zeijts
PRA, 45, 1163 (1992))

$$P = P_0 \, e^{-\left(\frac{x_{rms}}{x_{tol}}\right)^4}$$

$$x_{tol} = \begin{cases} \dfrac{L_S}{L_G} \times 0.145 \sqrt{\lambda_s L_G} \left(\dfrac{L_G}{z}\right)^{\frac{1}{4}} & (L_s \gg L_G) \\ \left(\dfrac{L_s}{L_G}\right)^{\frac{3}{4}} \times 0.266 \sqrt{\lambda_s L_G} \left(\dfrac{L_G}{z}\right)^{\frac{1}{4}} & (L_s \ll L_G) \end{cases}$$

L_s correction station spacing

z wiggler distance

Cornell wiggler

$L_G = 0.2636 \, m$

$L_S = 0.495 \, m \quad > L_G$

$L_W = z = 1.98 \, m$

$\lambda_s = 5 \mu m$

$$P = P_0 \, e^{-\left(\frac{L_G}{L_S}\right)^4 \left(\frac{x_{rms}}{0.145\sqrt{\lambda_s L_G}}\right)^4 \frac{z}{L_G}}$$

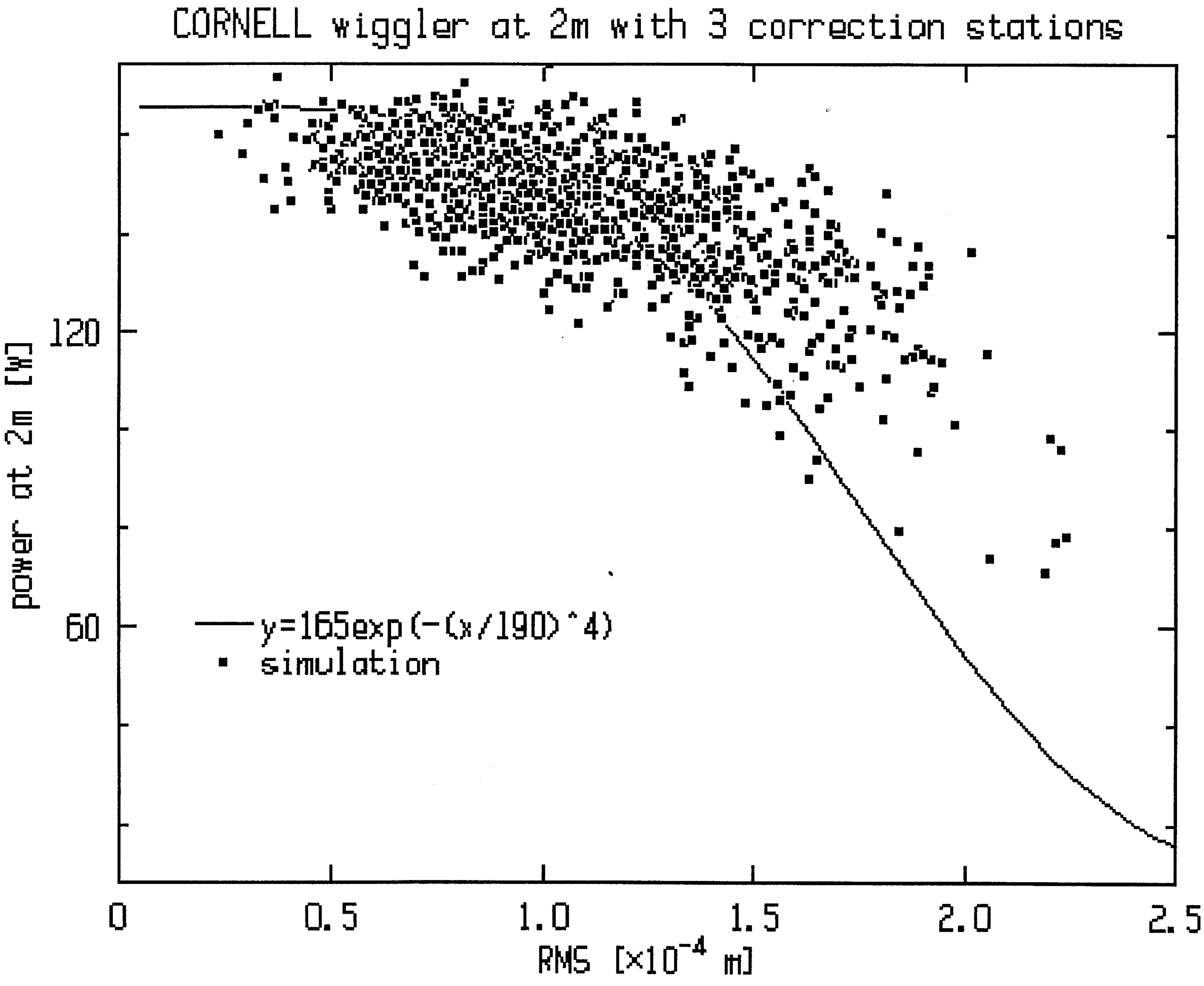
$$x_{tol} = \frac{L_S}{L_G} \times 0.145 \sqrt{\lambda_s L_G} \times \left(\frac{L_G}{z}\right)^{\frac{1}{4}}$$

$$= \frac{0.495}{0.264} \times 0.145 \times \sqrt{5 \times 10^{-6} \times 0.264} \times \left(\frac{0.264}{1.98}\right)^{\frac{1}{4}}$$

$$= 3.12 \times 10^{-4} \times 0.606 = 1.89 \times 10^{-4}$$

$\simeq 190 \mu m$

$$P = P_0 \, e^{-\left(\frac{x_{rms}}{190\mu}\right)^4}$$


CORNELL wiggler at 2m with 3 correction stations
power at 2m [W]
120
60
0
0.5
1.0
1.5
2.0
2.5
RMS [×10^-4 m]
y=165exp(-(x/190)^4)
simulation

LCLS

$L_s = 2.16\,m$

$L_G = 6.3\,m$ $\qquad (L_s < L_G)$

$\lambda_s = 1.5 \times 10^{-10}\,m$

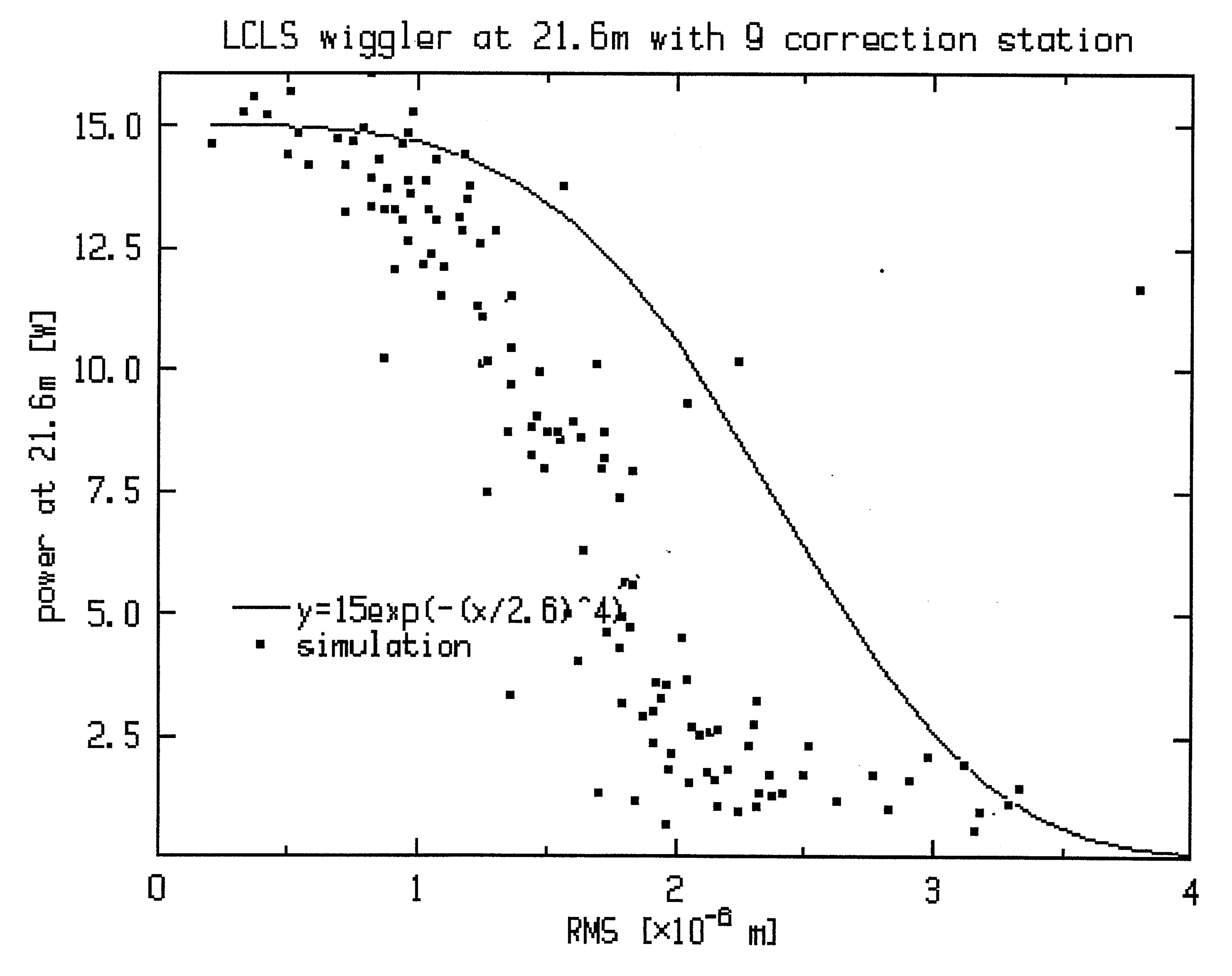
take $z = 21.6\,m$

$$P = P_0\, e^{-\left(\frac{L_G}{L_s}\right)^3 \left(\frac{x_{rms}}{0.266\sqrt{\lambda_s L_G}}\right)^4 \frac{z}{L_G}}$$

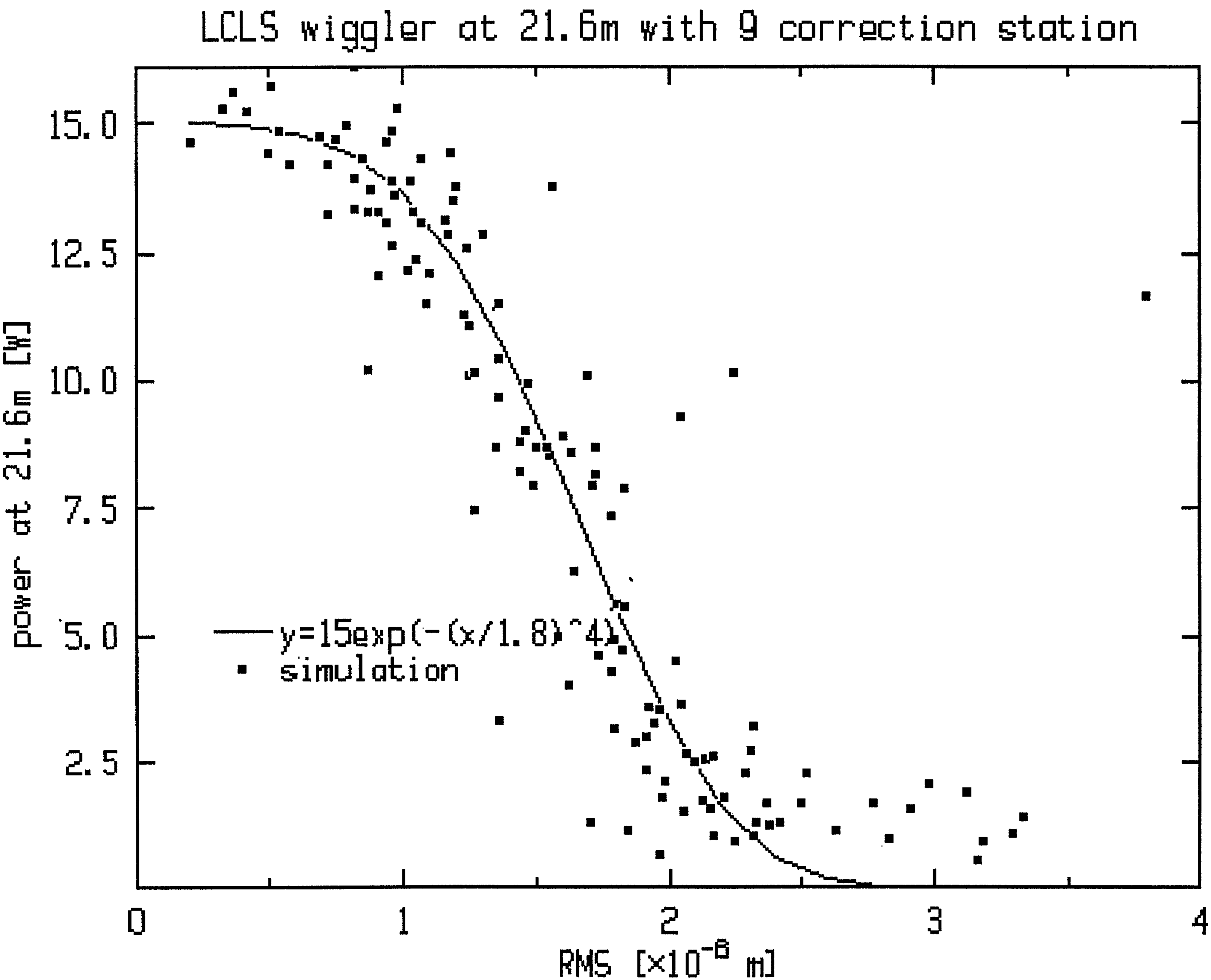
$$x_{tol} = \left(\frac{L_s}{L_G}\right)^{\frac{3}{4}} \times 0.266\sqrt{\lambda_s L_G} \times \left(\frac{L_G}{z}\right)^{\frac{1}{4}}$$

$$= 2.6\,\mu m$$

$$\therefore\ P = P_0\, e^{-\left(\frac{x_{rms}}{2.6\mu}\right)^4}$$



LCLS wiggler at 21.6m with 9 correction station
power at 21.6m [W]
15.0
12.5
10.0
7.5
5.0
2.5
0
1
2
3
4
RMS [×10^-6 m]
y=15exp(-(x/2.6)^4)
simulation



LCLS wiggler at 21.6m with 9 correction station
power at 21.6m [W]
15.0
12.5
10.0
7.5
5.0
2.5
0
1
2
3
4
RMS [×10^-6 m]
y=15exp(-(x/1.8)^4)
simulation

VISA 0.8μ

$L_G = 0.184\,m$

$L_s = 0.5\,m > L_G$

$L_w = z = 4\,m$

$\lambda_s = 0.8\,\mu m$

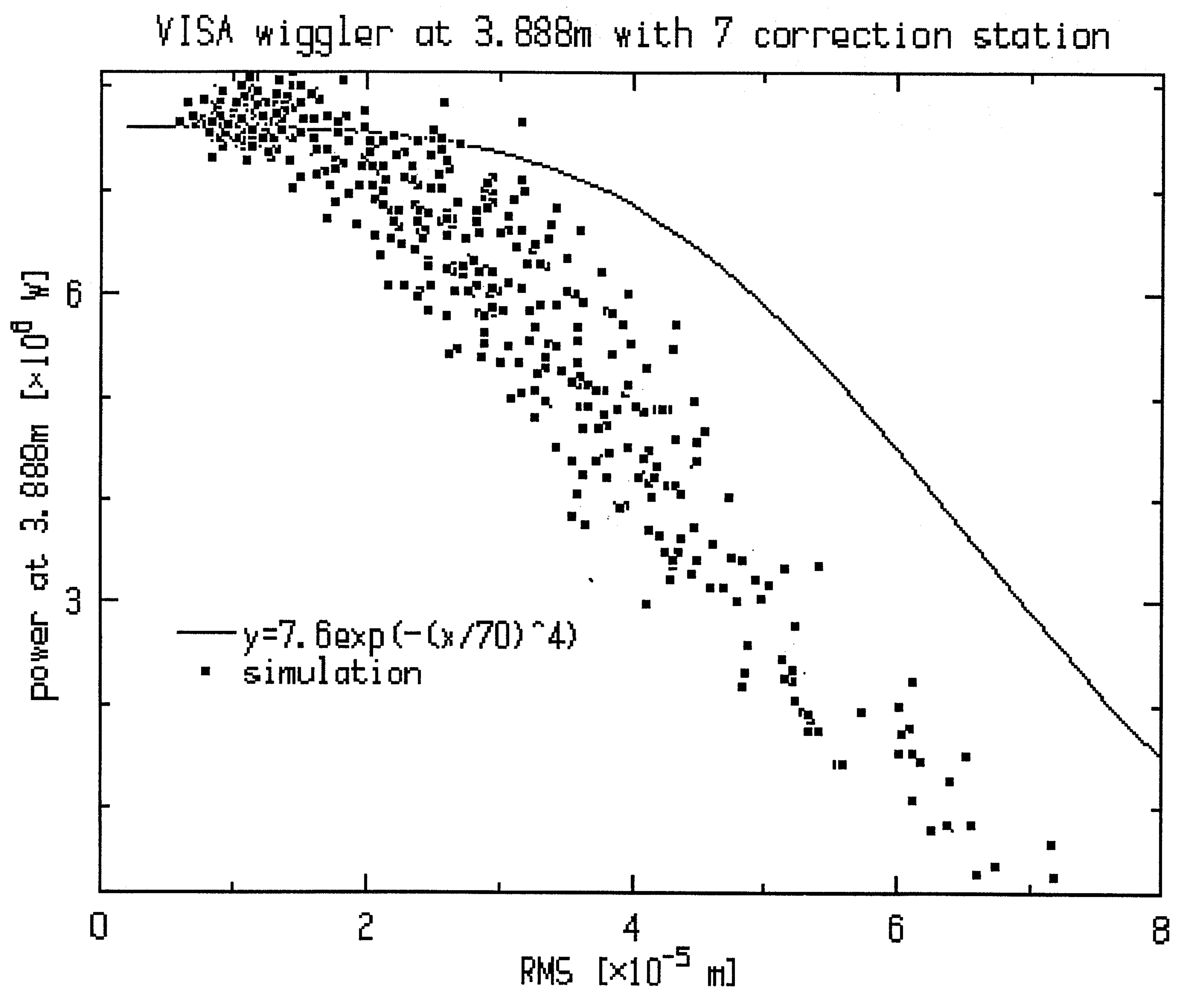
$$P = P_0\, e^{-\left(\frac{L_G}{L_s}\right)^4 \left(\frac{x_{rms}}{0.145\sqrt{\lambda_s L_G}}\right)^4 \frac{z}{L_G}}$$

$$\frac{L_s}{L_G} \times 0.145\sqrt{\lambda_s L_G} \times \left(\frac{L_G}{z}\right)^{\frac{1}{4}}$$

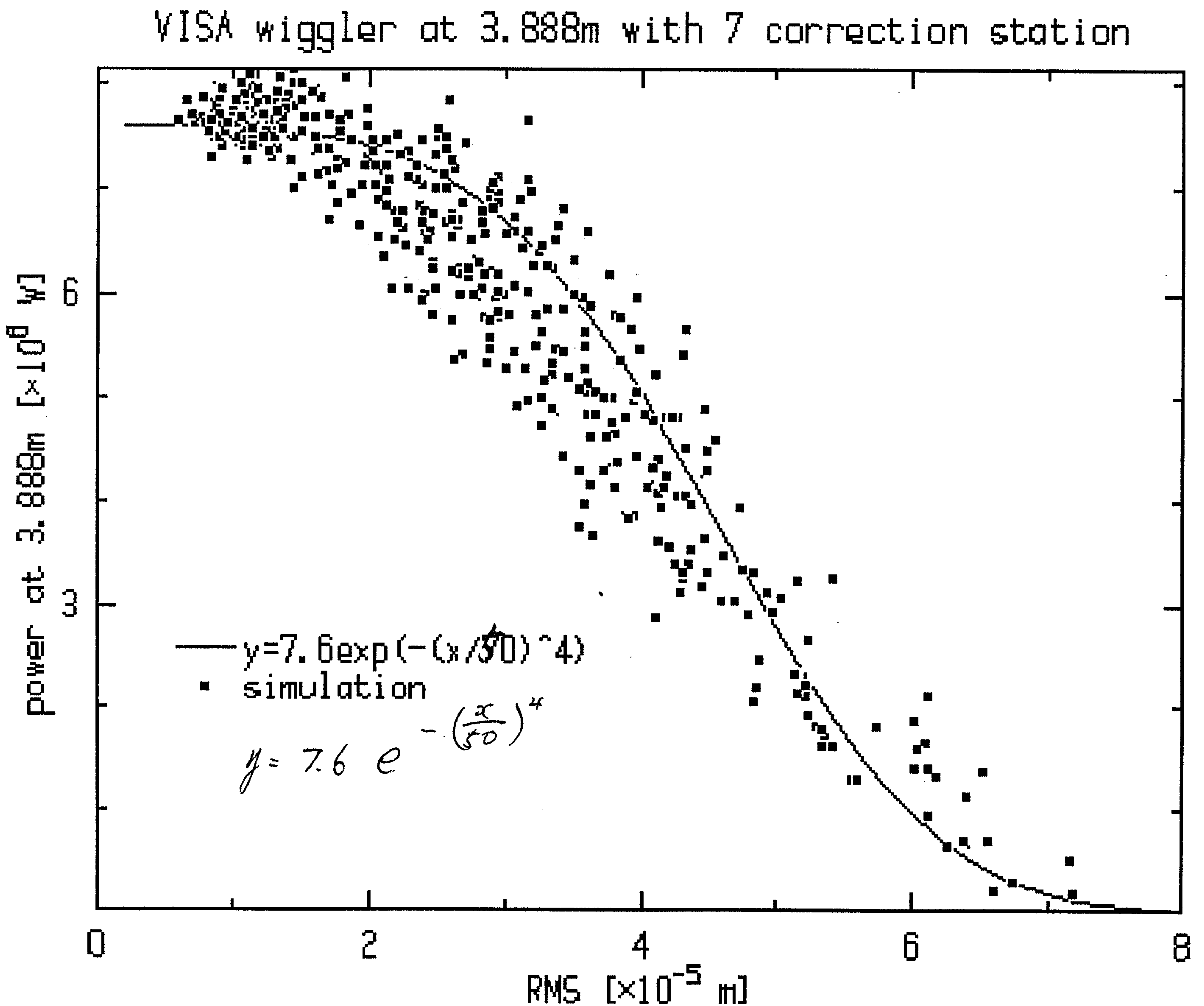
$$= \frac{0.5}{0.184} \times 0.145 \times \sqrt{0.8\times10^{-6}\times 0.184} \times \left(\frac{0.184}{4}\right)^{\frac{1}{4}}$$

$$= 1.5\times10^{-4} \times \left(\frac{0.184}{4}\right)^{\frac{1}{4}} = 0.67\times10^{-4} \simeq 70\mu$$

$$\therefore P = P_0\, e^{-\left(\frac{x_{rms}}{70\mu}\right)^4}$$



VISA wiggler at 3.888m with 7 correction station
power at 3.888m [×10^8 W]
6
3
0
2
4
6
8
RMS [×10^-5 m]
y=7.6exp(-(x/70)^4)
simulation



VISA wiggler at 3.888m with 7 correction station
power at 3.888m [x10^8 W]
RMS [x10^-5 m]
0
2
4
6
8
3
6
y=7.6exp(-(x/50)^4)
simulation
$y = 7.6\, e^{-\left(\frac{x}{50}\right)^4}$

3. SASE/spontaneous radiation ratio

$$D = \left[\frac{8 I_0}{I_A} \cdot \frac{K^2}{1+\frac{K^2}{2}} \cdot \frac{J^2}{\gamma} \right]^{\frac{1}{2}}$$

$$\rho_s = \left(\frac{D^2}{128\, k_s\, k_w\, \sigma_x^2} \right)^{\frac{1}{3}} \quad \text{(step function)}$$

$$\tilde{\hat{a}}_s = \frac{D}{2\sqrt{2}\,\rho_s}$$

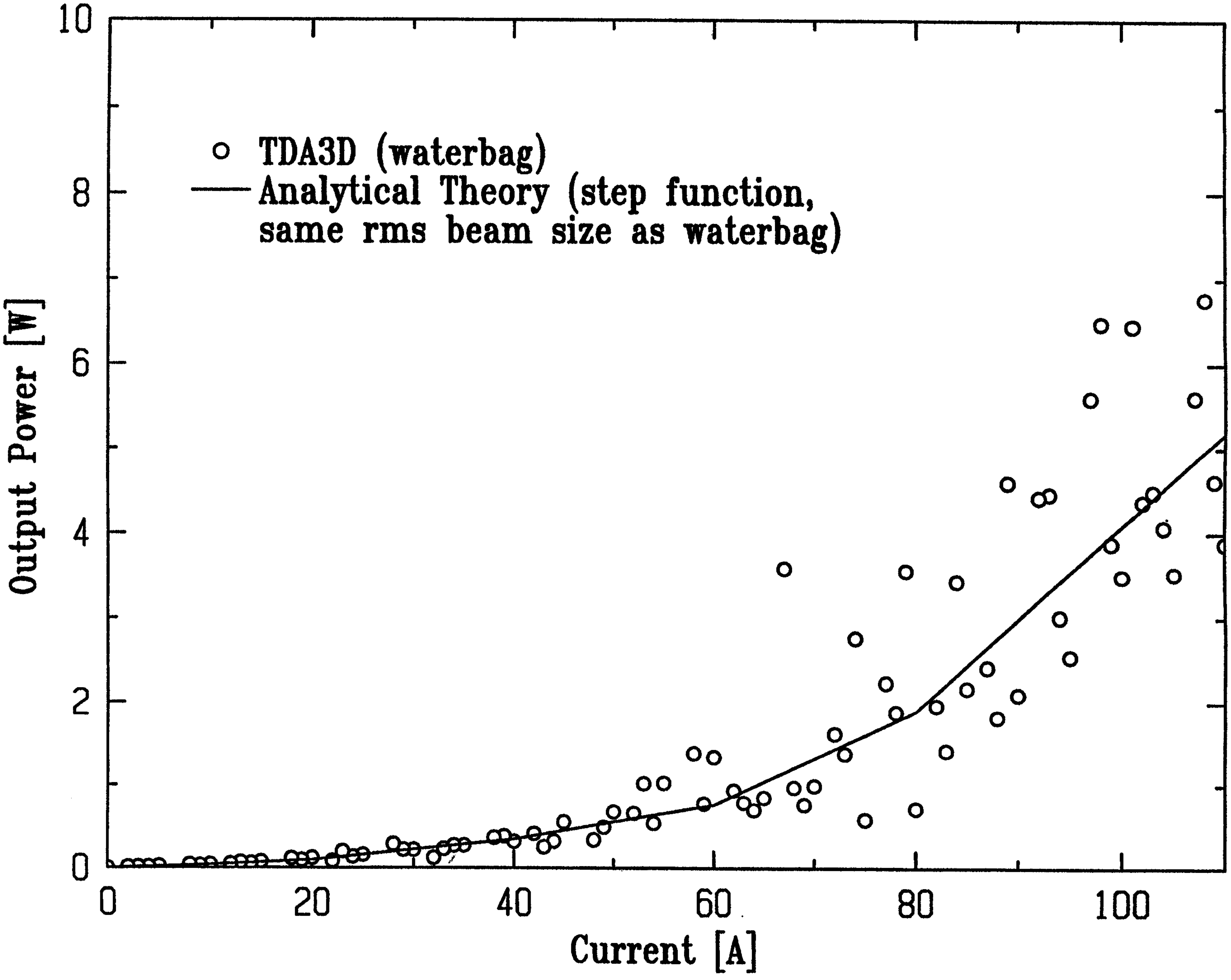
$$C_0(\tilde{\hat{a}}_s) \cong \frac{\sqrt{3}}{\pi \tilde{\hat{a}}_s^2}\, e^{-\frac{1}{\tilde{\hat{a}}_s \sqrt{1+\tilde{\hat{a}}_s^2}} \left(\beta_0 + \beta_1 \frac{1}{\tilde{\hat{a}}_s^2} \right)}$$

$$\beta_0 = 1.093, \ \beta_1 = -0.02 \ , \ (\tilde{\hat{a}}_s > 0.25)$$

$$\frac{\left(\frac{dP}{d\omega}\right)_{SASE}}{\left(\frac{dP}{d\omega}\right)_{spon}^{L_W}} - 1 = \left(\frac{1}{9}\, e^{\frac{L_W}{L_G}} - 1 \right) C_0(\tilde{\hat{a}}_s)\, \frac{2 L_G}{L_W} + \cdots$$

$$\sqrt{2\pi}\, \frac{\sigma_\omega}{\omega} = \frac{1}{N_W} \left(\frac{3}{4\pi} \cdot \frac{L_W}{L_G} \right)^{\frac{1}{2}}$$

$$P = \sqrt{2\pi}\, \frac{\sigma_\omega}{\omega} \left(\frac{dP}{d\omega} \right)$$



○ TDA3D (waterbag)
— Analytical Theory (step function, same rms beam size as waterbag)
Output Power [W]
Current [A]
0
2
4
6
8
10
0
20
40
60
80
100

fig. 3

Example: Cornell wiggler SASE (5μ)

take central peak:

$I_0 = 180$ amp

$\varepsilon_n = 4$ mm - mrad , $\lambda_s = 5\mu m$

$\gamma = 82$

$K = 1.44$ $\qquad \lambda_w = 3.3$ cm

$\lambda_\beta = 3.75$ m , $\sigma_x = 180\ \mu m$, $L_w = 2$ m

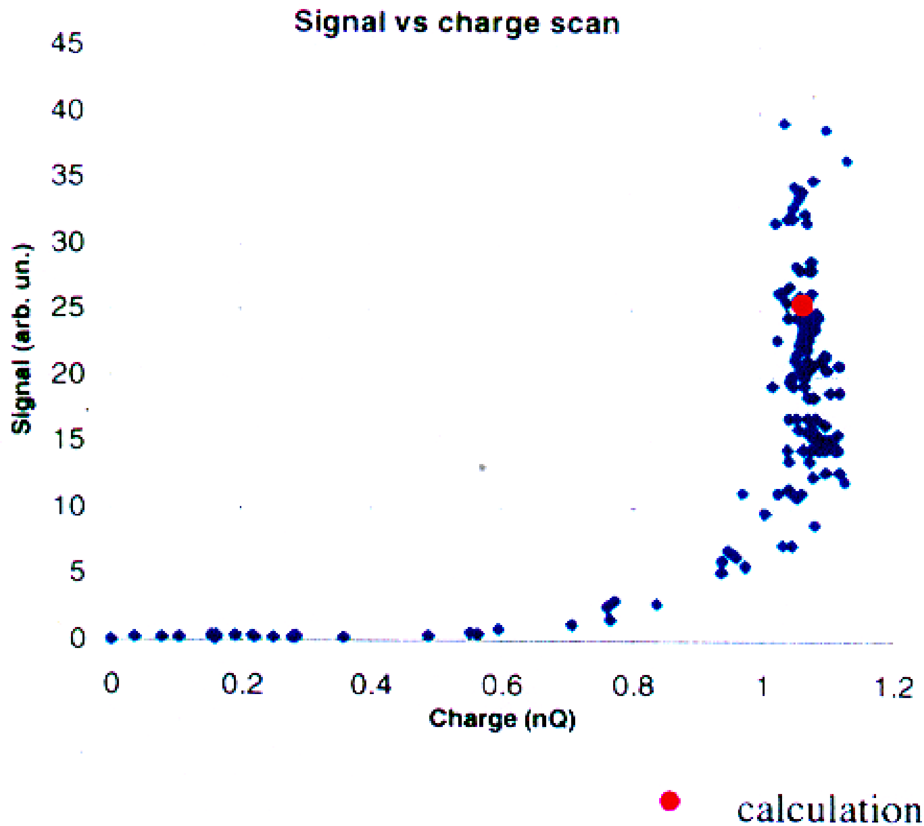
get

$D = 2.8 \times 10^{-2}$, $\rho_s = 9.23 \times 10^{-3}$

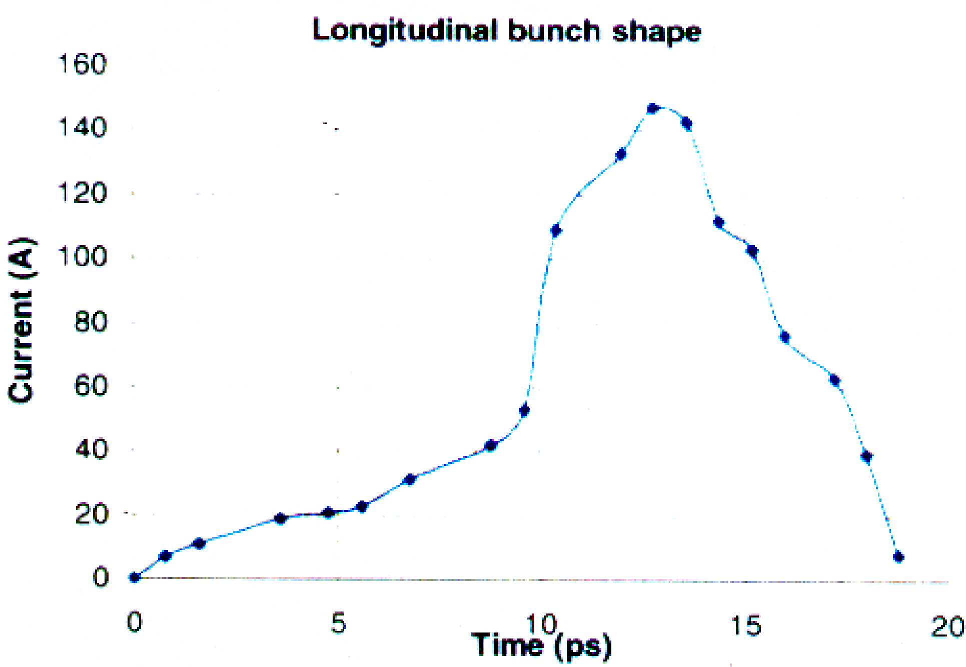
$\hat{\tilde{a}}_s = 1.08$, $\qquad C_0 = 0.24$

$I_m(\lambda) = 0.676$

$$\frac{\left(\frac{dP}{d\omega}\right)_{SASE}}{\left(\frac{dP}{d\omega}\right)^{L_w}_{spon}} - 1 = 69$$



Signal vs charge scan
Signal (arb. un.)
45
40
35
30
25
20
15
10
5
0
0
0.2
0.4
0.6
0.8
1
1.2
Charge (nQ)
calculation



Longitudinal bunch shape
Current (A)
160
140
120
100
80
60
40
20
0
0
5
10
15
20
Time (ps)

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3. Start-up calculation extended to 3D: large beam size limit, agrees with 1D
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