THE SCHERM SPACE CHARGE ROUTINE - LIMITATIONS AND SOLUTIONS

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Abstract

The SCHERM space charge routine reported in the 1996 linac conference is based on a representation of the charge density distribution in a bunch with an Hermite series expansion. Approximate values of the field components are deduced from the properties of the Hermite functions. First applications of the method for the 180 mA CERN proton linac showed promising results, however the method has shown its limitations when it was tested in a periodic accelerating channel (6 MeV to 100 MeV) with highly tune-depressed beams. Problems underlying the approach are discussed, and solutions are proposed.

1 INTRODUCTION

A new type of approach for the space charge computation, without the need of a strict symmetry has been reported in [1]. It is based on the 3-dimensional representation of the charge density distribution with a Hermite-series expansion. Approximated values of the field components were deduced in order to be good in the bunch core where most of the particles lie. The method had been successfully applied at the 180 mA CERN proton linac beam. Results compared very well with those computed with SCHEFF for this beam. Emboldened by the initial success, we applied this approach to a periodic channel with highly tune-depressed beam. This exercise has revealed underlying limitation of the method. These limits are found to be purely mathematical in origin and lie with the limitations of Hermite-series expansion of the charge density distribution and the field components. Solutions for circumventing these difficulties are proposed in this article.

2 PROBLEMS WITH SERIES REPRESENTATION

The difficulties with series representation had been reported at the end of the eighteenth century. Dirichlet had shown (~1870) that the Fourier-series expansion of some class of functions does not always converge uniformly (e.g. Gibbs phenomena). Moreover, when they converge uniformly, they oscillate around the function in a random manner. Using the studies of Cesaro on the divergent series, Fejer (~1890) presented solutions to such problems. Here, one represents, for simplification, the process in one dimension, which can be extended in 3 dimensions. Among the two representations of the m-order Hermite series expansion $S_m(x)$ of the function f(x), only the expression proposed in [1,3], converges to zero, when x goes to infinity, and as such can be used for the charge density distribution representation as:

$$S_m(x) = \sum_{n=0}^{m} A_n \cdot H_n(x) \cdot \exp\left(-\frac{x^2}{2}\right). \qquad (1)$$

For functions f without Gaussian appearance (meaning <u>continuous</u> functions decreasing to zero at the infinity more rapidly that any power of 1/|x|), S_m usually does not converge uniformly to f when m increases. However S_m always minimises the quadratic means square I_m :

$$I_m = \int_{-\infty}^{+\infty} \left(f(x) - S_m(x)\right)^2 \cdot dx . \qquad (2)$$

Due to eq. (2), the Hermite series in eq. (1) oscillates around the function f(x) (Weisstrass Minimax theorem). If S_m converges uniformly to f, these oscillations disappear when m increases to infinity. As the value of m is always limited, S_m can have a random behaviour with unexpected effects when the function f(x) changes during the computation. Moreover, if S_m does not converge uniformly to f, these oscillations are amplified with increasing m. This problem can be circumvented with the Cesaro-Fejer transformation. Here, one replaces the Hermite series $S_m(x)$ by the Cesaro-Fejer series $\sigma_m(x)$:

$$\sigma_m(x) = \frac{1}{m+1} \sum_{n=0}^{m} S_n(x). \qquad (3)$$

According to the Theorem of Cesaro, if the limit f(x) exists, the Cesaro-Fejer series in eq. (3), converges uniformly to f(x), when m increases. The unexpected oscillations are attenuated. An example can be seen in figure 1.

As presented in ref.[1], the local charge density expressed in terms of Hermite-series polynomial is given by:

$$\rho(x,y,z)=\exp\left(-\frac{u\ +v\ +w}{2}\right)\cdot \sum_i\sum_j\sum_k A_{ijk}\cdot H_i(u)\cdot H_j(v)\cdot H_k(w) \quad (4)$$

with :

$$A_{ijk}=\frac{q}{(2\pi)^{3/}\ i!\,j!\,k!\,abc}\cdot\sum_{n=1}^{N}H_i(u_n)\cdot H_j(v_n)\cdot H_k(w_n) \quad (5)$$

where u=x/a, v=y/b, w=z/c, a, b and c are the rms size of the bunch in the x, y and z directions respectively and N is the number of particles.

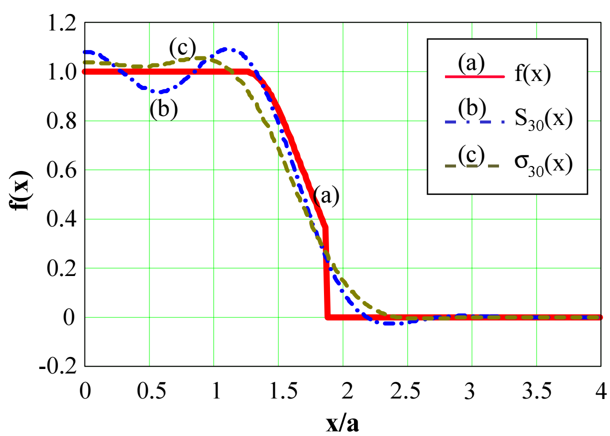

Figure 1: A function f(x) (in a) is represented by a 30-order Hermite series (in b). The oscillations practically disappears using a Cesaro-Fejer transformation (c).

The Cesaro-Fejer transformation of the one dimensional function in eq. (1) leads to :

$$\sigma_m(x)=\sum_{n=0}^{m}A_n\cdot\left(1-\frac{n}{m+1}\right)\cdot H_n(u)\cdot\exp\left(-\frac{u^2}{2}\right), \quad (6)$$

this can be extended to 3 dimensions.

From eq. (6), we see that Cesaro-Fejer transformation is an Hermite-expansion with attenuated high order coefficients. This plays the role of a low-pass filter in the Hermite base suppressing oscillations, but having a slope smaller than that of the Hermite-series.

The effect of the Cesaro-Fejer transformation can be seen in fig 2, where are presented the transverse emittance growth in a periodic accelerating channel with highly tune depressed beam (APT linac from 6.7 MeV up to 100 MeV). The computations are made for identical conditions with and without the Cesaro-Fejer transformation. It should be noted that the improvements in the field computation, explained below, have not yet been introduced. The difference in emittance growth, is due to the unexpected oscillations resulting from the Hermite series expansion of the charge density distribution. This oscillation of the charge density representation seems to introduce a diffusion like effect (or more precisely a stochastic coupling effect between directions) adding to the transverse emittance growth.

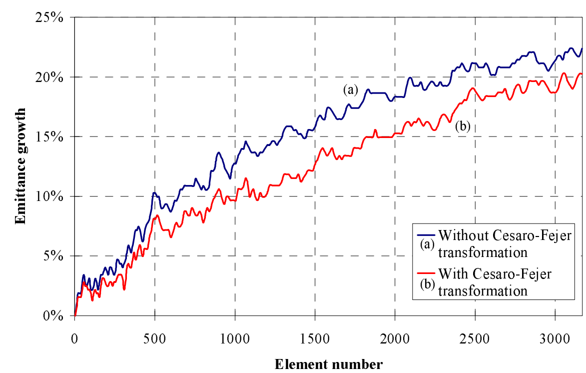

Figure 2 : Transverse emittance growth in a periodic accelerating channel (6.7 MeV to 100 MeV) with a highly tune-depressed beam. The results are obtained from the simple Hermite series expansion (a), and from the Cesaro-Fejer transformation (b).

3 SELECTION OF THE NUMBER OF TERMS IN THE HERMITE SERIES EXPANSION

It has been shown in [1], that the charge density distribution can be considered to be a Gaussian:

$$\rho_0(x,y,z)=A_{000}\exp\left(-\frac{u^2+v^2+w^2}{2}\right), \quad (7)$$

corrected by various terms (coefficients A_{ijk} in eq. (4)), each of them of total charge zero. In practice most of these coefficients can be neglected. It is essential to define a criterion, allowing the selection of the most significant terms. The Hermite polynomials $H_i(u)$ are obtained from the recurrence relation :

$$H_{i+1}(u)=u\cdot H_i(u)-i\cdot H_{i-1}(u). \quad (8)$$

Even and odd Hermite functions are represented in fig.3:

$$\mathcal{H}_i(u)=H_i(u)\cdot e^{-\frac{u^2}{2}}. \quad (9)$$

With even functions, the extremum is obtained when u=0. Near this point, from the Hermite recurrence relation, one obtains :

$$|H_{2l}(u)|\le(2l-1)\cdot(2l-3)\cdot\ldots\cdot 1. \quad (10)$$

For odd Hermite functions, the extremum lies between $0.5 < u < 0.75$, and one can write :

$$|H_{2l-1}(u)|\le(2l-2)\cdot(2l-4)\cdot\ldots\cdot 0.75. \quad (11)$$

Introducing the following numbers:

$$g(0) = 1$$
$$g(1) = 0.75 \tag{12}$$
$$g(n+2) = (n+1)\cdot g(n),$$

one obtains from eq. (10) and eq. (11) :

$$\left|A_{ijk} \cdot H_i(u)\cdot H_j(v)\cdot H_k(w)\right| \le \left|A_{ijk}\right| \cdot g(i)\cdot g(j)\cdot g(k). \tag{13}$$

Comparing the magnitude of the right term in eq. (13), relative to the Gaussian in eq. (7) allows us to select the more significant terms. They are chosen such that:

$$\left|A_{ijk}\right| \cdot g(i)\cdot g(j)\cdot g(k) \ge \delta \cdot A_{000}, \tag{14}$$

with δ = 0.1, 0.01, ..., depending on the accuracy desired. An estimation of the relative importance of the terms neglected can be obtained from eq. (14). The field contributions from only relatively more significant terms are considered.

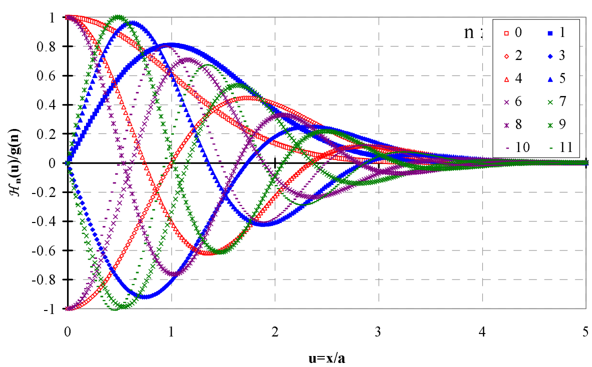

Figure 3 : The functions $H_n(u)/g(n)$ for n=0 to 11. They decrease rapidly to zero and might be considered null when u>5 (corresponding to 5 standard-deviation). The definition of g(n) is given in the text.

4 COMPUTATION OF THE FIELD COMPONENTS

Computation of the field components due to the dominant term A_{000} in eq. (7) is based on one quadrature calculation as explained in [2,3]. The field components due the other terms A_{ijk} are exactly calculated from the Poisson equation, instead of the approximated approach given in [1].

The corresponding potential from each term is given by the Poisson law :

$$\Delta U_{ijk}(u,v,w) = \frac{-A_{ijk}}{\varepsilon_0} \cdot \mathcal{H}_i(u)\cdot \mathcal{H}_j(v)\cdot \mathcal{H}_k(w) \tag{15}$$

Fourier transforms applied to eq. (15) leads to analytical expressions of the field components with a 3 dimensional (3D) integral in a 3D complex plane. Extensive effort was devoted to reduce the computation time of the field components.

The calculus of residues allows to reduce the 3D quadrature to a 2D one. This 2D quadrature is solved using numerical methods, which required several steps. Moreover, most of the values involved in these quadratures are independant of the particle co-ordinates and can be pre-calculated only once at the beginning of a simulation.

The analytical formulation of the method is quite extensive and beyond the scope of this paper.

5 CONCLUSION

Solutions to overcome the limits of the method presented in [1] are formulated. In particular:

- Cesaro-Fejer transformation would avoid the unexpected effects arising from the oscillations in the Hermite-series expansion, reducing the observed emittance growth.
- Criterion has been set (eq. (14)) to select the most significant terms of the series expansion. Computation have shown that only some tens of terms are sufficient for a suitable fit of the charge density distribution.
- Precision of field calculation has been improved and efforts have been done to reduce the computation time. Transformations applied to Poisson equation are being studied. Possibilities to reduce the 2D quadrature to a 1D one are also being looked out.

All the components of the above formualtion must be fully implemented in the space-charge code SCHERM before a fair beam-dynamics comparison could be done with DYNAC [4] and PARMILA [5] codes.

6 REFERENCES

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