OPTICS ELEMENTS FOR MODELING ELECTROSTATIC LENSES AND ACCELERATOR COMPONENTS IV. ELECTROSTATIC QUADRUPOLES AND SPACE CHARGE MODELING

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Abstract

Optical models for a variety of electrostatic elements have been developed for the computer code TRACE 3-D. TRACE 3-D is an envelope (matrix) code that includes a linear space charge model and is primarily used to model bunched beams in magnetic transport systems and radiofrequency (RF) accelerators. New matrix models have been developed that allow the code to be used for modeling beamlines and accelerators with electrostatic components. These new models include a number of options for simulating: (1) einzel lenses, (2) dc accelerator columns, (3) electrostatic deflectors (prisms), and (4) electrostatic quadrupoles. A prescription for setting up the initial beam appropriate to modeling 2-D (continuous) beams has also been developed. The models for (4) are described in this paper and examples of their use are illustrated. The relationship between the 3-D (bunched beam) and 2-D (dc beam) space charge modeling is discussed and comparisons of numerical results to other calculations are presented.

1 INTRODUCTION

The TRACE 3-D program [1] is one of the standard codes used in the design of standing wave radiofrequency linacs and transport lines for high-current bunched beams. Considerable work has been done on extending the program to model a new array of accelerator problems, including wakefields [2], traveling wave structures [3], and electrostatic lenses [4]. This paper describes recent work at further extending the capabilities of TRACE 3-D.

2 ELECTROSTATIC QUADRUPOLES

Two electrostatic (ES) quadrupoles models have been developed for use in TRACE 3-D. One is a hard-edge model where the magnitude of the quadrupole field is constant over the quadrupole length and zero elsewhere. The second models fringe fields as linear functions that act over specified fringe field entrance and exit distances.

2.1 Hard-Edge ES Quadrupole

The first order optics for a particle moving in the field of an ES quad are the same as those for the motion in a magnetic quad using an equivalent field gradient B':

$$B' = 2V_o/(a^2\beta c) \quad , \qquad (1)$$

where V_o is the electrode voltage of the ES quadrupole, a is the radial aperture of the ES quadrupole, and βc is the velocity of the particle. The hard-edge ES quad model [2] in TRACE 3-D simply calls the hard-edge magnetic quadrupole subroutine using a gradient given by (1).

2.2 ES Quadrupole with Fringe Field

Electrostatic quadrupoles with fringe fields are often modeled in terms of a potential function of the form:

$$(x,y,z) = + V(z)\,(x^2 - y^2)/a^2 \quad , \qquad (2)$$

where V(z) is a smooth function used to model the longitudinal variation of the quadrupole strength. The electric field is given by the gradient of ϕ: $\boldsymbol{E} = -\nabla\phi$. However, this electric field does not, in general, satisfy Maxwell's equation $\nabla\bullet\boldsymbol{E} = 0$. For the special case in which $V(z)$ is a piece-wise linear (or constant) function of z, then $\nabla\bullet\boldsymbol{E} = 0$ almost everywhere. The fringe field ES quadrupole uses a function $V(z)$ which rises linearly from zero to a maximum value V_o over an entrance length d_1, remains constant at V_o for a distance given by the effective electrode length l, and then decreases linearly to zero over an exit length d_2.

2.3 R-Matrix Elements and Example

A first-order 6×6 transfer matrix (R-matrix) is used to describe particle optics in the paraxial approximation. The elements of the R-matrix are computed directly from the electric fields using standard methods [4,5]. For the fringe field ES quadrupole, the region over which the fields act is divided into small steps of length Δz and four R-matrices are computed for each step: a drift matrix [1] of length $\Delta z/2$, a lens matrix which computes the quadrupole impulse, another drift matrix of length $\Delta z/2$, and a space-charge impulse matrix to model the linear space-charge forces (described in Section 3).

The non-trivial elements of the quadrupole lens R-matrix at location z are:

$$R_{21} = - R_{43} = - 2q\Delta z\, V(z)/(a^2\beta^2\gamma mc^2) \quad . \qquad (3)$$

Table 1 summarizes test TRACE 3-D calculations carried out for the hard-edge and fringe-field models with short fringe lengths, which are compared to magnetic quadrupole results. The fringe-field ES quad calculations required a maximum step size of $\Delta z = 0.1$ mm to obtain the results shown, whereas accurate results were obtained for the magnetic and hard-edge ES quads with $\Delta z = 2.0$ mm. Using the relation (1), the agreement is good between all cases.

Table 1. Comparison of quad fitting (matching) results (Example B of reference [1] without RF elements) for hard-edge ES quads, fringe-field ES quads (small d_1 and d_2), magnetic quads, and expected ES values from Eq. (1).

Quad Model	MMF (10^{-6})	Quad 1 (V_o or B')	Quad 2 (V_o or B')	Quad 3 (V_o or B')	Quad 4 (V_o or B')
Magnetic	2	- 20.2647 T/m	22.4726 T/m	- 19.6900 T/m	18.3534 T/m
Equation (1)	-	- 14.0189 kV/cm	15.5463 kV/cm	- 13.6213 kV/cm	12.6967 kV/cm
Hard-Edge ES	17	- 14.0189 kV/cm	15.5464 kV/cm	- 13.6214 kV/cm	12.6967 kV/cm
Fringe-Field ES	55	- 14.031 kV/cm	15.559 kV/cm	- 13.629 kV/cm	12.706 kV/cm

3 2-D SPACE CHARGE MODELING

Space charge is treated in TRACE 3-D as a linear force using the equivalent uniform beam model. The electric field components inside a uniformly charged 3-D ellipsoid, with semiaxes given by r_x, r_y and r_z, are [1]:

$$E_x(x,y,z) = (\kappa/\gamma^2)\,[(1-f(p))/(r_x(r_x+r_y)r_z)]x \quad , \qquad (4)$$
$$E_y(x,y,z) = (\kappa/\gamma^2)\,[(1-f(p))/(r_y(r_x+r_y)r_z)]y \quad , \qquad (5)$$

and

$$E_z(x,y,z) = \kappa\,[f(p)/(r_x r_y r_z)]z \quad , \qquad (6)$$

where $\kappa=3\lambda I_b/(4\pi\varepsilon_o c)$, γ is the relativistic energy factor of the beam, and f(p) is the 3-D ellipsoidal form factor. I_b is average beam current, where each bunch passes a given point once per RF cycle (wavelength is λ). The form factor depends on $p=\gamma r_z/(r_x r_y)^{1/2}$: for p>1, f(p) is:

$$f(p) = [p\,ln\,[p+(p^2+1)^{1/2}]/(p^2+1)^{3/2}] - 1/(p^2+1)^{1/2} \quad . \qquad (7)$$

Equations (4)-(6) for the bunched beam electric fields can simulate dc beam fields by taking appropriate limits.

3.1 Space Charge for Continuous Beams

For r_z much larger than r_x and r_y, the 3-D ellipsoidal beam bunch becomes elongated and the shape near the center approaches that of a 2-D beam with an elliptical cross section whose semiaxes are given by r_x and r_y. The electric fields in this case are obtained in the limit where p becomes very large. In the limit of large p, the 3-D ellipsoidal form factor becomes:

$$f(p) = [ln(2p) - 1] / p^2 \quad , \text{for } p>>1. \qquad (8)$$

The electric field becomes:

$$E_x(x,y,z) = (\kappa'/\gamma^2)\,[1/(r_x(r_x+r_y))]x \quad , \qquad (9)$$
$$E_y(x,y,z) = (\kappa'/\gamma^2)\,[1/(r_y(r_x+r_y))]y \quad , \qquad (10)$$
$$E_z(x,y,z) = (\kappa'/\gamma^2)\,[[ln(\gamma r_z/(r_x r_y)^{1/2}) - 1]/(r_z^2)]z \quad , \qquad (11)$$

where $\kappa'=(\kappa/r_z)$. The field components (9) and (10) are of the same form [8] as those for a continuous uniformly charged, 2-D elliptical cross section beam, with semiaxes r_x and r_y, when the parameter $\kappa'=I_{dc}/(\pi\varepsilon_o\beta c)$. I_{dc} is the current for the continuous (dc) beam. Consequently, the transverse electric fields computed by TRACE 3-D are the same as those for a dc beam if the bunched beam current is related to the continuous beam current by:

$$I_b = (4/3)(r_z/\beta\lambda)\,I_{dc} \quad . \qquad (12)$$

The longitudinal electric field (11) varies very slowly (logarithmically) with the transverse beam dimensions, and becomes small for large r_z. The bunch length r_z will not change due to the longitudinal space charge force if the total beamline length L over which the envelope equations are integrated is small compared to the initial r_z. Therefore, two conditions on the initial bunch length need to be satisfied so that the bunched beam space charge fields reduce to those for a continuous beam:

$$r_z >> (r_x r_y)^{1/2}/\gamma \quad \text{and } r_z > L \quad . \qquad (13)$$

Both conditions are achieved in the normal situation where the transverse dimensions are small compared to the beamline length, and one selects an initial value for the bunch length greater than L. The longitudinal emittance and Twiss parameters are:

$$\varepsilon_z = r_z(\Delta p/p) \quad \pi\text{-meter-radian} \quad , \qquad (14)$$
$$\alpha_z = 0 \quad , \qquad (15)$$

and

$$\beta_z = r_z/(\Delta p/p) \quad \text{meter/radian} \quad , \qquad (16)$$

where r_z is in meters and $\Delta p/p$ is the momentum spread.

Using the formulas (12) and (14)-(16), the TRACE 3-D space charge fields reduce to those for a 2-D continuous beam when the conditions (13) are satisfied.

3.2 Comparisons to Semi-Analytic Calculation

The accuracy of the 2-D simulation has been verified using TRACE 3-D by comparing the space charge radial expansion of cylindrical beams with results for the semi-analytic solutions. Table 2 summarizes one comparison.

The space charge expansion of a zero emittance, cylindrical beam can be expressed in terms of Dawson's integral. For a beam with radius r_o and no divergence at z=0, the downstream r and z are related by [7]:

$$(z/r_o) = (2/K)^{1/2}\,(r/r_o)\,D\,[ln((r/r_o)^{1/2}] \quad , \qquad (17)$$

where $D[\xi]$ is the value of Dawson's integral at ξ, and is available in tabulated form [9]. $K=2(I_{dc}/I_o)\,\beta^{-3}\gamma^{-3}$ is the generalized beam perveance and I_o is the Alfven current.

Table 2. Comparison of the simulated beam radius, for the space-charge expansion of 10 keV, 2 Ampere, dc (2-D uniform) e-beam, with the semi-analytic radius. The beam's initial phase space parameters are given in the text.

Length z (mm)	Drift Number	Drift Length	$\ln(r/r_o)^{1/2}$	Radius (mm) Semi-Analytic	Radius (mm) TRACE 3-D	Deviation (%)
1.6368	2	1.6368	0.02	10.00400	10.00400	0.00000
3.2750	3	1.6382	0.04	10.01601	10.01603	0.00020
8.2104	4	4.9354	0.10	10.10050	10.10057	0.00069
16.5869	5	8.3765	0.20	10.40811	10.40850	0.00375
119.6891	6	103.1022	1.00	27.18282	27.19945	0.06118
332.4853	7	212.7963	1.50	94.87736	94.94152	0.06763
548.5548	8	216.0694	1.70	179.93310	180.04664	0.06310
724.5571	9	176.0023	1.80	255.33722	255.49130	0.06034
976.9775	10	252.4205	1.90	369.66053	369.87306	0.05749

The Alfven current $I_o=4\pi\varepsilon_o[mc^3/q]=0.03335641 \times[mc^2(\mathrm{MeV})/q(e^-)]$ amps. The values $\xi=ln(r/r_o)^{1/2}$ shown in Table 2 were selected so that tabulated entries for D [ξ] could be used to determine the corresponding values of z/r_o. The perveance used (K=0.02986778) corresponds to a 2 amp, 10 keV beam with particle mass 0.511 MeV. A beamline of drift elements was constructed [10], whose lengths correspond to the intervals between the values of z/r_o. For a beam radius r_o = 10 mm, the resulting drift lengths and accumulated length z are given in Table 2.

The initial transverse phase space used values of $\varepsilon_x=\varepsilon_y=0.04$ π-mm-mrad, $\beta_x=\beta_y=2500$ mm/mrad and $\alpha_x=\alpha_y=0$. For the longitudinal parameters (14)-(16), initial values of $\Delta p/p = 5\times10^{-4}$ and r_z=100 meters were used. Simulations using I_b=0 confirmed that no beam expansion occurred due to finite transverse emittances.

The TRACE 3-D results shown in Table 2 used a radiofrequency of 2.998 MHz (λ just under 100 meters). Then, from Eq. (12), I_b=13.676 amps. The TRACE 3-D radii computed for this current agree with those from the semi-analytic calculation to better than 7 parts in 10,000. Other simulations to confirm the Eq. (12) scaling, and to explore limitations imposed by (13), were also performed with different λ, r_z, I_b, and with equal perveance beams.

4 SUMMARY

Optical elements for electrostatic quadrupoles have been developed for use in the TRACE 3-D code. A prescription for using TRACE 3-D to accurately simulate dc space charge effects has also been developed. Together with einzel lens [4], acceleration column [5], and deflector prism [6] models, TRACE 3-D has been expanded to model a spectrum of electrostatic systems.

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REFERENCES

[1] K. Crandall and D. Rusthoi, "TRACE 3-D Documentation," third edition, Los Alamos National Laboratory Report LA-UR-97-886 (1997) 106 pages.

[2] W. P. Lysenko, D. P. Rusthoi, K. C. D. Chan, G. H. Gillespie and B. W. Hill, "Wakefields in the TRACE 3-D Code," Proc. XVIII International Linear Accelerator Conf. (Geneva), Vol. 2, 845-847 (1996).

[3] M. C. Lampel, "A Self-Consistent Beam Loaded Traveling Wave Accelerator Model for Use in TRACE 3-D," to be published in the Proc. 1997 Particle Accelerator Conf. (Vancouver), 3 pages.

[4] G. H. Gillespie and T. A. Brown, "Optics Elements for Modeling Electrostatic Lenses and Accelerator Components I. Einzel Lenses," to be published in the Proc. 1997 Particle Accelerator Conf., 3 pages.

[5] G. H. Gillespie and T. A. Brown, "Optics Elements for Modeling Electrostatic Lenses and Accelerator Components II. Acceleration Columns," to be published in Nuclear Instruments and Methods in Physics Research B (1998) 4 pages.

[6] G. H. Gillespie, "Optics Elements for Modeling Electrostatic Lenses and Accelerator Components III. Deflector Prisms," in preparation.

[7] M. Reiser, *Theory and Design of Charged Particle Beams*, J. Wiley & Sons, New York (1994), p. 199.

[8] D. C. Carey, *The Optics of Charged Particle Beams*, Harwood Academic Publishers (1987), p. 284.

[9] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Ninth Printing, Dover Publications, New York (1972), p. 319, Table 7.5.

[10] *PowerTrace*™ is available from AccelSoft Inc., www.ghga.com/accelsoft.